Lagrange Equations in Classical Mechanics

Classical Mechanics

(1)

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The principle of least action or HAMILTON'S PRINCIPLE:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt \quad = \text{action} \sim [E]\cdot[T]$$

$L =$ lagrangian $\sim [E]$

Equations of motion ($q = q(t)$) are derived by requiring that $S = S_{min} \Rightarrow \delta S = 0$

Consider path variations where $q(t) \to q'(t) = q(t) + \delta q(t)$ and $\delta q_i(t_1) = \delta q_i(t_2) = 0$ (A)

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = \int_{t_1}^{t_2} dt \sum_i \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right\} = 0 \Rightarrow$$

$$\delta S = \int_{t_1}^{t_2} dt \sum_i \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) \right\} = 0 \Rightarrow$$

$$\delta S = \int_{t_1}^{t_2} dt \sum_i \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \cdot (-1) \right\} = 0 \Rightarrow$$

$$\delta S = \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right\} \delta q_i = 0$$

(first term $\to 0$ by (A))

$$\therefore \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \quad \text{Lagrange Equations}$$

Examples:

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Free particle:

$$L = \frac{1}{2} m \dot{x}^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad (\text{momentum}); \quad \frac{\partial L}{\partial x} = 0$$

$$\therefore \frac{d}{dt}(m\dot{x}) = 0 \Rightarrow m\ddot{x} = 0$$

Harmonic Oscillator:

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 = T - V$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}; \quad \frac{\partial L}{\partial x} = -kx \Rightarrow$$

$$\frac{d}{dt}(m\dot{x}) - (-kx) = 0 \Rightarrow \boxed{m\ddot{x} + kx = 0} \quad \text{H.O. Equation}$$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\omega^2 = k/m$$

Conservation Laws in Classical Mechanics: Energy and Momentum Conservation

Conservation Laws (3)

Symmetries of the Lagrangian of a system result to conserved quantities ⇒ Conservation Laws

Homogeneity of time: "The Lagrangian of a closed system cannot depend explicitly on time"

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \Rightarrow$$

(with $\frac{\partial L}{\partial t} = 0$)

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \Rightarrow$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \Rightarrow$$

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = 0$$

$$\frac{dH}{dt} = 0$$

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \quad \text{(Hamiltonian)}$$

Energy is conserved!!!

Homogeneity in Space: The mechanical properties of a closed system are unchanged under: (4)

$$\vec{r} \to \vec{r}\,' = \vec{r} + \vec{\epsilon}$$

$$\delta\vec{r} = \vec{\epsilon}$$

$$\delta L = \sum_a \frac{\partial L}{\partial \vec{r}_a}\delta\vec{r}_a = \vec{\epsilon}\cdot\sum_a \frac{\partial L}{\partial \vec{r}_a}$$

$$\delta L = 0 \to \sum_a \frac{\partial L}{\partial \vec{r}_a}\delta\vec{r}_a = \vec{\epsilon}\cdot\sum_a \frac{\partial L}{\partial \vec{r}_a} = 0 \Rightarrow$$

$$\sum_a \frac{\partial L}{\partial \vec{r}_a} = 0 \overset{\text{L.E}}{\Rightarrow} \sum_a \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_a}\right) = 0$$

$$\therefore \frac{d}{dt}\left(\sum_a \frac{\partial L}{\partial \vec{v}_a}\right) = 0$$

Momentum is Conserved !!

Conservation Laws in Classical Mechanics
Angular Momentum Conservation

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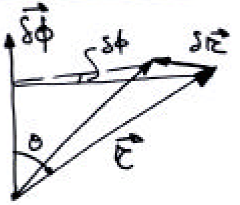
Isotropy in Space: The mechanical properties of a closed system do not vary when it is rotated as a whole.



$\hat{e}\,|\delta\vec{r}| = \delta\phi \cdot r \cdot \sin\theta\, \hat{e} = \vec{\delta\phi} \times \vec{r}$

$$\delta\vec{r} = \vec{\delta\phi} \times \vec{r}$$
$$\delta\vec{v} = \vec{\delta\phi} \times \vec{v}$$

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \vec{r}_a} \cdot \delta\vec{r}_a + \frac{\partial L}{\partial \vec{v}_a} \delta\vec{v}_a \right) = 0$$

$$\delta L = \sum_a \left(\dot{\vec{p}}_a \delta\vec{r}_a + \vec{p}_a \cdot \delta\vec{v}_a \right)$$

$$\delta L = \sum_a \left(\dot{\vec{p}}_a \cdot \vec{\delta\phi} \times \vec{r}_a + \vec{p}_a \cdot \vec{\delta\phi} \times \vec{v}_a \right)$$

$$\delta L = \sum_a \left(\vec{\delta\phi} \cdot (\vec{r}_a \times \dot{\vec{p}}_a) + \vec{\delta\phi} \cdot (\vec{v}_a \times \vec{p}_a) \right)$$

$$\delta L = \vec{\delta\phi} \cdot \sum_a \vec{r}_a \times \dot{\vec{p}}_a + \dot{\vec{r}}_a \times \vec{p}_a$$

$$\delta L = \vec{\delta\phi} \cdot \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a$$

} Angular momentum is conserved

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Summary:

1. Equations of motion are derived by the Lagrange equations acting on scalar Lagrangians
2. Symmetries of the Lagrangian lead to conservation laws

Introduction to Covariant Notation

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Lecture II: Quantum Field Theory (1)

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Review of covariant notation:

4-Vectors: A point in space-time can be defined by a 4-vector $x^\mu = (x^0, \vec{x})$ where $\mu = 0,1,2,3$ and $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$x^\mu(\tau) = (x^0(\tau), \vec{x}(\tau))$ describes a trajectory of an object in space time. The quantity $(x^0)^2 - \vec{x}^2$ is invariant under Lorentz transformations.

- Define CONTRAVARIANT 4-vectors as $X^\mu = (x^0, \vec{x})$ and COVARIANT 4-vectors as $X_\mu = (x^0, -\vec{x})$
- Define the dot product of two 4-vectors a^μ, b^μ as $a^\mu b_\mu = a^0 b^0 - \vec{a}\cdot\vec{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$

  This product is invariant under Lorentz transformations
- Define a metric as $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

  You can think then of a dot product as

$$(a^0, a^1, a^2, a^3) \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

Using the metric one can see that the dot product can be written as

$$a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \vec{a}\cdot\vec{b}$$

↑ Einstein's Convention

"Any index that appears twice once as a subscript and once as a superscript is understood to be summed"

- You can convert from covariant to contravariant vectors using the metric:

$$X^\mu = g^{\mu\nu} X_\nu \quad \text{or}$$
$$X_\mu = g_{\mu\nu} X^\nu$$

but $X^\mu X_\mu = g^{\mu\alpha} X_\alpha g_{\mu\beta} X^\beta \Rightarrow$

$$g^{\mu\alpha} g_{\mu\beta} = \delta^\alpha{}_\beta = \begin{cases} +1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

- Under Lorentz transformations, 4-vectors transform as:

$$\left.\begin{aligned} X'^\mu &= \Lambda^\mu{}_\nu X^\nu \\ X'_\mu &= \Lambda_\mu{}^\alpha X_\alpha \end{aligned}\right\} \Rightarrow X'_\mu \cdot X'^\mu = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta X_\alpha X^\beta$$

$$\Rightarrow \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \delta^\alpha{}_\beta$$

Therefore $\Lambda_\mu{}^\alpha$ is the inverse of $\Lambda^\mu{}_\beta$

Maxwell's Equation in covariant notation

④

Derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec\nabla\right)$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x^0}, -\vec\nabla\right)$$

and $$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec\nabla^2 = \Box$$

- So one can write the wave equation in a compact form as $\partial_\mu \partial^\mu \psi = 0$ or $\Box\psi = 0$
- Consider now the Maxwell equations

$$\nabla\cdot\vec E = 4\pi\rho$$
$$\vec\nabla\cdot\vec B = 0$$
$$\vec\nabla\times\vec E = -\frac{1}{c}\frac{\partial\vec B}{\partial t}$$
$$\vec\nabla\times\vec B = \frac{4\pi}{c}\vec J + \frac{1}{c}\frac{\partial\vec E}{\partial t}$$

recall that:

$$\vec E = -\vec\nabla\Phi - \frac{1}{c}\frac{\partial\vec A}{\partial t}$$
$$\vec B = \vec\nabla\times\vec A$$

Define a second-rank tensor which is antisym and is defined as $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$A^\mu = (\Phi, \vec A)$$

Under Lorentz $F'^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu}$

So $$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

as before one can convert to $F_{\alpha\beta}$ as

$$F_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

And $$\left.\begin{aligned} \vec\nabla\cdot\vec E &= 4\pi\rho \\ \vec\nabla\times\vec B &= \frac{4\pi}{c}\vec J + \frac{1}{c}\frac{\partial\vec E}{\partial t} \end{aligned}\right\} \Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$$

also $$\partial_\mu \partial_\nu F^{\mu\nu} = 0 = \frac{4\pi}{c}\partial_\nu J^\nu$$

$$\therefore \boxed{\partial_\nu J^\nu = 0}$$

Define also $$\tilde F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$$

where $$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \alpha=0, \beta=1, \gamma=2, \delta=3 \oplus \text{even perm.} \\ -1 & \text{any odd perm.} \\ 0 & \text{if any to indices are equal} \end{cases}$$

$$\left.\begin{aligned} &\vec\nabla\cdot\vec B \\ &\vec\nabla\times\vec E = \frac{1}{c}\frac{\partial\vec B}{\partial t} \end{aligned}\right\} \Rightarrow \boxed{\partial_\alpha \tilde F^{\alpha\beta} = 0}$$

Action and Lagrangian in Field Theory

Classical Field Theory (5)

- Consider the action S

$$S=\int d^4x\, \mathcal{L}(\phi(x),\partial_\mu\phi(x)) \quad \text{or}$$

$$S=\int dx^0\, L=\int dx^0\int d^3x\, \mathcal{L}(\phi(x),\partial_\mu\phi(x))$$

- $\mathcal{L}$ is called Lagrangian density
- L is the Lagrangian
- $\phi(x)$ is a field which in Quantum Field theory will be an operator that can create or destroy a particle

The field equations for $\phi(x)$ can be obtained from S by requiring that $\delta S=0 \Rightarrow S=S_{min}$ when one considers $\phi\to\phi'=\phi+\delta\phi$

The action S must be:

(a) Lorentz invariant + invariant under translations that is Poincaré invariant

(b) function of the fields and their derivatives (translational invariance again)

(c) depends on the fields taken at one space-time point x^μ only, leading to a local field theory (6)

(d) S must be a real # $\Rightarrow$ probability is conserved. An action which is complex leads to absorption i.e matter disappears into nothing $\Rightarrow$ no good

(e) must lead to classical equations of motion that involve no-higher than second-order derivatives $\Rightarrow$ we want a theory that has causal solutions

- If S is a number then

$$d^4x \sim L^4 \sim m^{-4} \Rightarrow \mathcal{L}\sim L^{-4}\sim m^4$$

So a Lagrangian density of the form

$$\mathcal{L}(x)=\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-\frac{m^2}{2}\phi^2$$

is a good guess

The Lagrange Equations in Field Theory

Lagrange Equations in Field Theory (7)

$$\left. \begin{array}{l} S = \int d^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\ \delta S = 0 \end{array} \right\} \Rightarrow$$

$$\int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \underbrace{\delta(\partial_\mu \phi)}_{\partial_\mu(\delta\phi)} \right\} = 0$$

$$\int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) - \delta\phi \, \partial_\mu \left(\frac{\partial L}{\partial \partial_\mu \phi} \right) \right\} = 0$$

$$\int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \right\} \delta\phi + \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta\phi = 0$$

SURFACE TERM → 0

$$\therefore \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) - \frac{\partial L}{\partial \phi} = 0}$$

Recall $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$ in Classical mechanics

Example: Scalar field (8)

$$\left. \begin{array}{l} \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \\ \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) - \frac{\partial L}{\partial \phi} = 0 \end{array} \right\} \Rightarrow$$

$$\frac{\partial}{\partial x^\mu} \left\{ \frac{1}{2} \delta^\mu{}_\alpha \partial^\alpha \phi + \frac{1}{2} \partial_\alpha \phi \, g^{\mu\alpha} \right\} + \frac{m^2}{2} 2\phi = 0$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

$$(\partial_\mu \partial^\mu + m^2) \phi(x) = 0 \quad \text{Klein-Gordon Equation}$$

or $\quad (\Box + m^2) \phi(x) = 0$

The Vector Field Lagrangian

Example: The Electromagnetic Field

$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Rightarrow$ Lorentz scalar

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow F_{\mu\nu} = -F_{\nu\mu}$

$\mathcal{L} = \frac{1}{4} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) F^{\alpha\beta} = \frac{1}{4}(\partial_\alpha A_\beta F^{\alpha\beta} - \partial_\beta A_\alpha F^{\alpha\beta})$

$\mathcal{L} = \frac{1}{4} (\partial_\alpha A_\beta F^{\alpha\beta} + \partial_\beta A_\alpha F^{\beta\alpha}) = \frac{1}{2} \partial_\alpha A_\beta F^{\alpha\beta}$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = \frac{\partial}{\partial \partial_\mu A_\nu} \left\{ \frac{1}{2} \partial_\alpha A_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \right\}$$

$$= \left\{ \frac{1}{2} \delta^\mu{}_\alpha \delta^\nu{}_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + 1/2\, \partial_\alpha A_\beta (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) \right\}$$

$$= \frac{1}{2} \left\{ \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu A^\nu - \partial^\nu A^\mu \right\}$$

$$= F^{\mu\nu} \quad ; \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$\partial_\mu F^{\mu\nu} = 0$ Maxwell equation no current

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(9) Introduce the Field current coupling

$$\mathcal{L}_I = \frac{4\pi}{c} J_\mu A^\mu$$

(later it will come as a result of U(1) local gauge theory)

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_I = +\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{4\pi}{c} J_\mu A^\mu$$

Then $\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{4\pi}{c} J_\alpha g^{\nu\alpha} = J^\nu \cdot \frac{4\pi}{c}$

$$\therefore \quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

The Spin 1/2 Lagrangian and the Dirac Equation

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Pauli Matrices:

$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

PROPERTIES: $\sigma_i^2 = 1$

$\sigma_i \sigma_j = \delta_{ij} + i\,\epsilon_{ijk}\sigma_k$

$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

$[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}$

$\{1, \sigma_1, \sigma_2, \sigma_3\}$ is a basis in the 2x2 matrix space

Dirac Matrices:

$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$; $\gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$; $\gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$\{\gamma^\mu, \gamma^5\} = 0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

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The Dirac Field

Define a 4-component spinor $\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

and $\bar\psi = \psi^\dagger\gamma^0$ that is

$\bar\psi_\alpha = \psi^\dagger_\beta(\gamma^0)_{\beta\alpha}$

Next consider the Lagrangian:

$\mathcal{L}(x) = \frac{i}{2}\left\{\bar\psi\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar\psi)\gamma^\mu\psi\right\} - m\bar\psi\psi$

$\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar\psi_\alpha} = -\frac{i}{2}(\gamma^\mu)_{\alpha\beta}\psi_\beta$

$\frac{\partial\mathcal{L}}{\partial\bar\psi_\alpha} = \frac{i}{2}(\gamma^\mu)_{\alpha\beta}\partial_\mu\psi_\beta - m\psi_\alpha$

$\therefore \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar\psi_\alpha}\right) - \frac{\partial\mathcal{L}}{\partial\bar\psi_\alpha} = 0 \Rightarrow$

$-\frac{i}{2}\gamma^\mu_{\alpha\beta}\partial_\mu\psi_\beta + -\frac{i}{2}\gamma^\mu_{\alpha\beta}\partial_\mu\psi_\beta + m\psi_\alpha = 0$

$-i\partial\!\!\!/\psi + m\psi = 0$

$(i\partial\!\!\!/ - m)\psi = 0$ Dirac's Equation

The Dirac Equation

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or $\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_\alpha} = \frac{i}{2} \bar{\psi}_\beta \gamma^\mu_{\beta\alpha}$

$$\frac{\partial \mathcal{L}}{\partial \psi_\alpha} = -\frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu - m \bar{\psi}$$

$$\therefore \partial_\mu \left(\tfrac{i}{2} \bar{\psi} \gamma^\mu \right) + \frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0 \Rightarrow$$

$$i \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0 \Rightarrow$$

$$\bar{\psi} (i \overleftarrow{\partial\!\!\!/} + m) = 0$$

just another way to write Dirac's equation for $\bar{\psi}$

Symmetries and Conservation Laws

Symmetries and Conservation Laws (I)

Consider the Poincare transformation:

$$x^{\mu'} = x^\mu + \omega^{\mu\nu} x_\nu + a^\mu \quad \text{or}$$

$$\delta x^\mu = \omega^{\mu\nu} x_\nu + a^\mu \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

In general a field which could be a scalar or a vector or a spinor would transform as:

$$\Phi^{a'}(x') = \left[\delta^a{}_\beta - \tfrac{1}{2}\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\right]\Phi^\beta(x')$$

If the field is scalar then $\Sigma^a{}_\beta{}^{\mu\nu} = 0$

a, β are "field indices" which could be space time indices in the case of a vector field, but they could also be $a, b = 1,2,3,4$ in the case of a spinor.

$$\delta\phi^a = \phi^a(x') - \tfrac{1}{2}\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x') - \phi^a(x)$$

$$\delta\phi^a = \Phi^a(x+\omega\cdot x+a) - \Phi^a(x) - 1/2\,\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x')$$

$$\delta\Phi^a = a^\mu\partial_\mu\Phi^a(x) + \omega^{\mu\nu}x_\nu\partial_\mu\Phi^a(x) - \tfrac{1}{2}\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x)$$

(higher order terms dropped)

So:

$$\boxed{\delta\phi^a = a^\mu\partial_\mu\Phi^a(x) + \tfrac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\Phi^a(x) + {} - 1/2\,\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x)}$$

(II)

Under Poincare transformations the Lagrangian $\mathcal{L}$ should be a scalar that is

$$\mathcal{L}\left(\phi'(x'), \frac{\partial\phi'(x')}{\partial x_\mu}\right) = \mathcal{L}\left(\phi(x'), \frac{\partial\Phi(x')}{x'_\mu}\right)$$

(functional form remains the same)

Therefore:

$$\delta x^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\delta(\partial_\mu\phi^a) + \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\delta\phi^a\right] - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\right]\delta\phi^a + \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a$$

(by Lagrange equation $= 0$)

$$\therefore\ \delta x^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\delta\phi^a\right\} \quad \text{"Emmy Noether"}$$

$$(\omega^{\mu\nu}x_\nu + a^\mu)\partial_\mu\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\left(a^\rho\partial_\rho\phi^a + \tfrac{1}{2}\omega^{\rho\sigma}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi^a\right.\right.$$

$$\left.\left. - 1/2\,\omega_{\rho\sigma}\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\right)\right\} \Rightarrow$$

The Energy Momentum Tensor

(III)

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}a^\rho\partial_\rho\phi^a\right] - a^\mu\partial_\mu\mathcal{L} - \omega^{\mu\nu}x_\nu\partial_\mu\mathcal{L} +$$

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\frac{1}{2}\omega^{\rho\sigma}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi^a\right] +$$

$$-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\left(\frac{1}{2}\omega_{\rho\sigma}\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\right)\right] = 0$$

$$a^\rho\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial_\rho\Phi^a - g^\mu{}_\rho\mathcal{L}\right] - \frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\mathcal{L}$$

$$+\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\frac{1}{2}\omega^{\rho\sigma}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi^a\right] +$$

$$-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\frac{1}{2}\omega_{\rho\sigma}\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\right] = 0$$

But $$\boxed{T^{\mu\rho} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial^\rho\phi^a - g^{\mu\rho}\mathcal{L}}$$ (Energy momentum tensor)

and $$f^{\mu\rho\sigma} = 1/2\,\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}$$ (some tensor)

Therefore:

$$a_\rho\partial_\mu T^{\mu\rho} - \frac{1}{2}\omega^{\mu\nu}\partial_\rho\left[\delta^\rho{}_\mu x_\nu\mathcal{L} - x_\mu\delta^\rho{}_\nu\mathcal{L}\right] +$$

$$-1/2\,\omega_{\rho\sigma}\partial_\mu\left(f^{\mu\rho\sigma} - f^{\mu\sigma\rho}\right) + \frac{1}{2}\omega^{\rho\sigma}\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi^a\right\} = 0$$

(IV)

Rename indices:

$$a_\rho\partial_\mu T^{\mu\rho} - \frac{1}{2}\omega^{\rho\sigma}\partial_\mu\left[\delta^\mu{}_\rho x_\sigma\mathcal{L} - x_\rho\delta^\mu{}_\sigma\mathcal{L}\right] +$$

$$-\frac{1}{2}\omega_{\rho\sigma}\partial_\mu\left(f^{\mu\rho\sigma} - f^{\mu\sigma\rho}\right) +$$

$$1/2\,\omega^{\rho\sigma}\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi^a\right] = 0$$

$\Rightarrow$

$$a_\rho\partial_\mu T^{\mu\rho} + \frac{1}{2}\omega^{\rho\sigma}\partial_\mu\Big\{x_\rho\delta^\mu{}_\sigma\mathcal{L} - x_\sigma\delta^\mu{}_\rho\mathcal{L} +$$

$$f^{\mu\sigma\rho} - f^{\mu\rho\sigma} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi^a\Big\} = 0$$

Define
$$M^\mu{}_{\rho\sigma} = f^{\mu\sigma\rho} - f^{\mu\rho\sigma} + x_\sigma\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial_\rho\phi^a - \delta^\mu{}_\rho\mathcal{L}\right) - x_\rho\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial_\sigma\phi^a - \delta^\mu{}_\sigma\mathcal{L}\right)$$

That is $$\boxed{M^\mu{}_{\rho\sigma} = f^{\mu\sigma\rho} - f^{\mu\rho\sigma} + x_\sigma T^\mu{}_\rho - x_\rho T^\mu{}_\sigma}$$

Internal Symmetries of the Lagrangian

(V)

In Conclusion:

$$\alpha_\rho \partial_\mu T^{\mu\rho} + \frac{1}{2}\omega^{\rho\sigma}\partial_\mu M^\mu{}_{\rho\sigma} = 0$$

If this is true for any α_ρ and $\omega_{\rho\sigma}$ (any Poincare transformation) then it must be:

$$\partial_\mu T^{\mu\rho} = 0 \quad \text{(Conservation of Energy-Momentum)}$$

$$\partial_\mu M^\mu{}_{\rho\sigma} = 0 \quad \text{(Angular momentum is conserved)}$$

$$T^{\mu\rho} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial^\rho\phi^a - g^{\mu\rho}\mathcal{L}$$

$$f^{\mu\rho\sigma} = \frac{1}{2}\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta \cdot \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}$$

$$M^{\mu\rho\sigma} = f^{\mu\sigma\rho} - f^{\mu\rho\sigma} + x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma}$$

And

$$\Sigma^\alpha{}_\beta{}^{\rho\sigma} = \begin{cases} 0 & \text{for scalar fields} \\ g^{\alpha\rho}g_\beta{}^\sigma - g^{\alpha\sigma}g_\beta{}^\rho & \text{for vector fields} \\ 1/4\,[\gamma^\rho,\gamma^\sigma]^\alpha{}_\beta & \text{Spinors} \end{cases}$$

INTERNAL SYMMETRIES

(VI)

Often a Lagrangian is invariant under transformations of the type:

$$\Phi(x) \to e^{-i\epsilon\lambda}\Phi(x) \cong 1 - i\epsilon\lambda\cdot\Phi$$

Φ in general is $\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{pmatrix}$ and λ is $n\times n$ matrix:

$$\Phi_i(x) \to \Phi_i(x) - i\epsilon\lambda_{ij}\Phi_j(x)$$

The transformations form a group and λ_{ij} are the group generators

If one constructs a Lagrangian which is invariant under these transformations then we have

$$\delta\mathcal{L} = 0 = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta(\partial_\mu\phi_i)$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right] + \underbrace{\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right)\delta\phi_i}_{=0 \text{ using Lagrange Equations}}$$

Enough for today……..

VII

So

$$\delta \mathcal{L} = 0 = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \phi_i \right\}$$

$$\boxed{\partial_\mu J^\mu = 0}$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \phi_i \Rightarrow$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} (-i\epsilon) \lambda_{ij} \phi_j(x)$$

$$\boxed{J^\mu = -i\epsilon \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \lambda_{ij} \phi_j(x)}$$

So internal symmetries lead to conserved currents:

$$\partial_0 J^0 = 0 \Rightarrow J^0 = -i\epsilon \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \lambda_{ij} \phi_j$$

$$J^0 = -i\epsilon \pi_i \lambda_{ij} \phi_j$$

$$Q = -i \int d^3x \, \pi_i \lambda_{ij} \phi_j$$

$$dQ/dt = 0$$

Expanding the scalar field in terms of creation and annihilation operators

(14)

The next step now is to quantize the classical fields we studied so far.

Back to the scalar field $\Phi(x)$

Let: $\Phi(x) = \int \frac{d^4k}{(2\pi)^{3/2}} e^{-ik\cdot x}\,\tilde\varphi(k)$ (F.T.)

where $k\cdot x = k^0x^0 - \vec{k}\cdot\vec{x}$

$$(\Box + m^2)\Phi(x) = 0 \Rightarrow$$

$$\int \frac{d^4k}{(2\pi)^{3/2}} \underbrace{(\partial_\mu\partial^\mu + m^2)\, e^{-ik\cdot x}}_{\{(-ik)\cdot(-ik)+m^2\}e^{-ik\cdot x}}\,\tilde\varphi(k) = 0$$

$$\therefore (k^2 - m^2)\,\tilde\varphi(k) = 0$$

Try $\tilde\varphi(k) = \delta(k^2 - m^2)\,\varphi(k)$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{-ik\cdot x}\,\delta(k^2 - m^2)\,\varphi(k)$$

remember $\delta(\vec{k} - \vec{k}') = \frac{1}{(2\pi)^3}\int e^{-i\vec{x}(\vec{k}-\vec{k}')}\,d^3x$

$$\delta(x^2 - a^2) = \frac{1}{2a}\{\delta(x-a) + \delta(x+a)\}$$

(15)

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}}\int e^{-i(k^0x^0 - \vec{k}\vec{x})}\,\delta(k^{0\,2} - \vec{k}^2 - m^2)\,\varphi(k)\,dk^0 d^3\vec{k}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}}\int e^{-i(k^0x^0 - \vec{k}\vec{x})}\,\varphi(k)\,\frac{1}{2\sqrt{\vec{k}^2+m^2}}\left\{\delta(k^0 - \sqrt{\vec{k}^2+m^2}) + \delta(k^0 + \sqrt{\vec{k}^2+m^2})\right\}dk^0 d^3k$$

Call $E_k = \sqrt{\vec{k}^2+m^2}$ and

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k \int dk^0\, \varphi(k)\,\frac{1}{2E_k}\{\delta(k^0 - E_k) + \delta(k^0 + E_k)\}\,e^{-ik\cdot x}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{2E_k}\Big\{\underbrace{\varphi(E_k,\vec{k})}_{\varphi_+}\, e^{-iE_kx^0 + i\vec{k}\vec{x}} + \underbrace{\varphi(-E_k,\vec{k})}_{\varphi_-}\, e^{iE_kx^0 + i\vec{k}\vec{x}}\Big\}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{2E_k}\Big[\varphi_+(\vec{k})\, e^{-iE_kx^0 + i\vec{k}\vec{x}} + \varphi_-(\vec{k})\, e^{iE_kx^0 + i\vec{k}\vec{x}}\Big] \quad (1)$$

Since we have already in mind that $\Phi(x)$ will become an operator in Quantum Field Theory, we demand that it is Hermitian that is $\Phi^\dagger(x) = \Phi(x) \Rightarrow$

$$\underbrace{\varphi_+^\dagger(\vec{k})\, e^{iE_kx^0 - i\vec{k}\vec{x}}}\; + \;\varphi_-^\dagger(\vec{k})\, e^{-iE_kx^0 - i\vec{k}\vec{x}} =$$

$$\varphi_+(\vec{k})\, e^{-iE_kx^0 + i\vec{k}\vec{x}} + \underbrace{\varphi_-(\vec{k})\, e^{iE_kx^0 + i\vec{k}\vec{x}}}$$

$$\varphi_+^\dagger(\vec{k}) = \varphi_-(-\vec{k}) \Rightarrow \boxed{\varphi_-(\vec{k}) = \varphi_+^\dagger(-\vec{k})} \quad (2)$$

The Quantization of the Scalar Field I

①③ →

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k}\left[\varphi_+(\vec{k})\, e^{-iE_k x^0 + i\vec{k}\vec{x}} + \underbrace{\varphi_+(-\vec{k})\, e^{iE_k x^0 + i\vec{k}\vec{x}}}_{\varphi_+^\dagger(\vec{k})\, e^{iE_k x^0 - i\vec{k}\vec{x}}}\right] \quad (16)$$

define $a_{\vec{k}} = \frac{\varphi_+(\vec{k})}{\sqrt{2E_k}}$

$$\therefore \Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{(2E_k)^{1/2}} \left\{ a_{\vec{k}}\, e^{-iE_k x^0 + i\vec{k}\vec{x}} + a^\dagger_{\vec{k}}\, e^{iE_k x^0 - i\vec{k}\vec{x}} \right\}$$

Scalar Field Quantization

1._ Identify canonical variables and replace them with operators. In Quantum Field Theory all fields are operators.

2._ Impose canonical commutation relations (similar to $[\hat p, \hat x] = -i\hbar$ in Quantum Mechanics)

3._ The demand that the field obeys position-momentum-type of commutation relations results to a Quantized Field (Quantum Field theory)

(17)

At this point recall the theory of the harmonic oscillator in Quantum Mechanics since the procedure to quantize a field is very similar.

Start with $L = \frac{\hat p^2}{2m} - \frac{1}{2} m\omega^2 \hat q^2 \Rightarrow H = p\dot q - L \Rightarrow$

$$\boxed{\hat H = \frac{\hat p^2}{2m} + \frac{1}{2} m\omega^2 \hat q^2} \quad (1)$$

Define $\hat a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat q + \frac{i\hat p}{m\omega}\right)$ and $\hat a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat q - \frac{i\hat p}{m\omega}\right)$

Demand that $[\hat q, \hat p] = i\hbar$ (2)

(1)(2) $\Rightarrow$ $H = \hbar\omega\left(\hat a^\dagger \hat a + \frac{1}{2}\right)$ (3)

$[\hat q, \hat p] = i\hbar \Leftrightarrow [a, a^\dagger] = 1$ (4)

So by defining a Lagrangian and from it a Hamiltonian we have defined a classical theory. The requirement $[\hat q, \hat p] = i\hbar$ makes it a Quantum theory because:

Say $H|S\rangle = E_s|S\rangle$, what is $a|S\rangle$? is it also eigenstate of H

$$H a|S\rangle = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) a|S\rangle = \frac{\hbar\omega}{2} a|S\rangle + \hbar\omega a^\dagger a a|S\rangle$$

$$H a|S\rangle = \frac{\hbar\omega}{2} a|S\rangle + \hbar\omega (a a^\dagger - 1) a|S\rangle$$

$$H a|S\rangle = \frac{\hbar\omega}{2} a|S\rangle + \hbar\omega a a^\dagger a|S\rangle - \hbar\omega a|S\rangle$$

$$H a|S\rangle = a H|S\rangle - \hbar\omega a|S\rangle = (E_s - \hbar\omega) a|S\rangle \;!$$

The Quantization of the Scalar Field II

(18)

So $\hat{a}|S\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue $E_S - \hbar\omega$

$\hat{a} \rightarrow$ "lowering operator" (annihilation)

$\hat{a}^{\dagger} \rightarrow$ "raising operator" (creation)

Back to the scalar field $\Phi(x)$

Define $\pi(\vec{x},t) = \partial^0 \Phi(\vec{x},t)$ (momentum?)

and require $[\Phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$

to quantize the field.

$$\Phi(\vec{x}',t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k'}{\sqrt{2E_k'}} \left\{ e^{-iE_k'x^0 + i\vec{k}'\cdot\vec{x}'} a_{k'} + a_{k'}^{\dagger} e^{iE_k'x^0 - i\vec{k}'\vec{x}'} \right\}$$

$$\dot{\Phi} = \pi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} \left\{ (-i) e^{-iE_k x^0 + i\vec{k}\vec{x}} (E_k) + a_k^{\dagger} (iE_k) e^{iE_k x^0 - i\vec{k}\vec{x}} \right\}$$

$$[\Phi(\vec{x}',t), \pi(\vec{x},t)] = \frac{1}{(2\pi)^3} \int \frac{d^3k'}{\sqrt{2E_k'}} \int \frac{d^3k}{\sqrt{2E_k}} \cdot \Big\{$$

$$\left\{ e^{-iE_k'x^0 + i\vec{k}'\vec{x}'} (-i) E_k \, e^{-iE_k x^0 + i\vec{k}\cdot\vec{x}} [a_{k'}, a_k] \right\}_{\text{kill}} +$$

$$\left\{ e^{-iE_k'x^0 + i\vec{k}'\vec{x}'} (iE_k) \, e^{iE_k x^0 - i\vec{k}\vec{x}} [a_{k'}, a_k^{\dagger}] \right\}_{\text{keep}} +$$

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$$+ \left\{ e^{iE_k'x^0 - i\vec{k}'\vec{x}'} e^{-iE_k x^0 + i\vec{k}\cdot\vec{x}} (-iE_k) [a_{k'}^{\dagger}, a_k] \right\}_{\text{keep}} +$$

$$\left\{ e^{iE_k'x^0 - i\vec{k}'\vec{x}'} (iE_k) \, e^{iE_k x^0 - i\vec{k}\vec{x}} [a_{k'}^{\dagger}, a_k^{\dagger}] \right\}_{\text{kill}} \Big\} =$$

$$= i\,\delta(\vec{x}-\vec{x}')$$

To do this we must require

$$[a_{k'}, a_k] = 0$$

$$[a_{k'}^{\dagger}, a_k^{\dagger}] = 0$$

$$[a_{k'}, a_k^{\dagger}] = \delta(\vec{k}'-\vec{k})$$

remember that $\delta(\vec{k}-\vec{k}') = \frac{1}{(2\pi)^3} \int d^3x \, e^{+i\vec{x}\cdot(\vec{k}-\vec{k}')}$

So
$$[\Phi(\vec{x}',t), \pi(\vec{x},t)] = \frac{1}{(2\pi)^3} \frac{i}{2} \int d^3k \, e^{i\vec{k}(\vec{x}-\vec{x}')} + \frac{1}{(2\pi)^3} \frac{i}{2} \int d^3k \, e^{i\vec{k}(\vec{x}'-\vec{x})}$$

$$= i\,\delta(\vec{x}-\vec{x}')$$

The Hamiltonian of the Scalar Field

The Hamiltonian of the scalar field (20) is:

$$H = \int d^3x \, \Pi(\vec{x},t)\, \partial^0 \phi(\vec{x},t) - L$$

$$L = \int d^3x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2 \right)$$

$$H = \frac{1}{2} \int d^3x \left\{ \Pi^2 + (\vec{\nabla}\phi)^2 + m^2 \Phi^2 \right\}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} \left[e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k + e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right]$$

$$\Pi(x) = \frac{-i}{(2\pi)^{3/2}} \int d^3k \sqrt{\frac{E_k}{2}} \left[e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k - e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right]$$

$$\vec{\nabla}\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} (i\vec{k}) \left[e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k - e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right]$$

(21)

$$H = \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3x \Big\{ (-i)^2 \int d^3k \sqrt{\frac{E_k}{2}} \int d^3k' \sqrt{\frac{E_{k'}}{2}} \times$$

$$\Big\{ \left(e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k - e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right) \cdot \left(e^{i\vec{k}'\cdot\vec{x} - iE_{k'} t} a_{k'} - e^{-i\vec{k}'\cdot\vec{x} + iE_{k'} t} a_{k'}^\dagger \right) \Big\} +$$

$$+ i^2 \int \frac{d^3k}{\sqrt{2E_k}} \int \frac{d^3k'}{\sqrt{2E_{k'}}} (\vec{k}\cdot\vec{k}') \times$$

$$\Big\{ \left(e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k - e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right) \cdot \left(e^{i\vec{k}'\cdot\vec{x} - iE_{k'} t} a_{k'} - e^{-i\vec{k}'\cdot\vec{x} + iE_{k'} t} a_{k'}^\dagger \right) \Big\} +$$

$$+ m^2 \int \frac{d^3k}{\sqrt{2E_k}} \int \frac{d^3k'}{\sqrt{2E_{k'}}} \left(e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k + e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right) \cdot \left(e^{i\vec{k}'\cdot\vec{x} - iE_{k'} t} a_{k'} + e^{-i\vec{k}'\cdot\vec{x} + iE_{k'} t} a_{k'}^\dagger \right)$$

The Hamiltonian with Creation and annihilation operators

(22)

$\Rightarrow$

$$H = \frac{1}{2}\times\frac{1}{2}\int d^3k \Big\{ a_{\vec{k}} a_{-\vec{k}} \Big[i^2 E_k - i^2 \frac{\vec{k}^2}{E_k} + \frac{m^2}{E_k} \Big] e^{2iE_k t} +$$

$$-E_k + \frac{k^2+m^2}{E_k} = 0$$

$$- a_{\vec{k}} a_{\vec{k}}^{\dagger} \Big[(-i)^2 E_k + i^2 \frac{\vec{k}^2}{E_k} - \frac{m^2}{E_k} \Big] +$$

$$-E_k - \frac{k^2+m^2}{E_k} = -2E_k$$

$$- a_k^{\dagger} a_{\vec{k}} \Big[(-i)^2 E_k + i^2 \frac{\vec{k}^2}{E_k} - \frac{m^2}{E_k} \Big]$$

$$-E_k - \frac{\vec{k}^2+m^2}{E_k} = -2E_k$$

$$+ a_{\vec{k}}^{\dagger} a_{-\vec{k}}^{\dagger} \Big[(-i)^2 E_k - 1\, i^2 \frac{\vec{k}^2}{E_k} + \frac{m^2}{E_k} \Big]$$

$$-E_k + \frac{\vec{k}^2+m^2}{E_k} = 0$$

$$H = \frac{1}{4}\int d^3k \Big\{ 2E_k a_{\vec{k}} a_{\vec{k}}^{\dagger} + 2E_k a_k^{\dagger} a_k \Big\}$$

(23)

$$H = \frac{1}{4}\int d^3k \Big\{ 2E_k a_k a_k^{\dagger} + 2E_k a_k^{\dagger} a_k \Big\}$$

$$H = \frac{1}{2}\int d^3k \Big\{ a_{\vec{k}} a_{\vec{k}}^{\dagger} + a_{\vec{k}}^{\dagger} a_k \Big\} E_k$$

$$[a_{\vec{k}}, a_{\vec{k}'}^{\dagger}] = \delta(\vec{k}-\vec{k}')$$

So $H = \frac{1}{2}\int d^3k\, E_{\vec{k}} \left(2 a_{\vec{k}}^{\dagger} a_k + \delta(0)\right)$

$$H = \frac{1}{2}\,2\int d^3k\, E_{\vec{k}}\, a_{\vec{k}}^{\dagger} a_k + \frac{1}{2}\int d^3k\, E_k\, \delta(0)$$

the infinite sum of the $\frac{\hbar\omega}{2}$ of all H.O.

NOT MEASURED

So I IGNORE

So

$$H = \int d^3k\, E_{\vec{k}}\, a_{\vec{k}}^{\dagger} a_{\vec{k}}$$

$$H|s\rangle = E_s|s\rangle$$

$$H a_k|s\rangle = \int d^3k' E_{k'} a_{k'}^{\dagger} a_{k'}\, a_k|s\rangle = \int d^3k' E_{k'} (a_k a_{k'}^{\dagger} - \delta(k-k'))|s\rangle$$

$$= \int d^3k' E_{k'}\, a_k a_{k'}^{\dagger} a_{k'}|s\rangle - E_k \hat{a}_k|s\rangle$$

$$= a_k H|s\rangle - E_k a_k|s\rangle$$

$$= (E_s - E_k)\, a_k|s\rangle$$ It lowers the energy by a quantum E_k

Summary: Quantum Field Theory

SUMMARY (24)

1. A Field theory is defined by a Lagrangian which is a function of the fields and the derivatives of the fields
2. $\delta S = 0$ gives the field equations
3. $[\pi(\vec{x},t), \phi(\vec{x}',t)] = -i\delta(\vec{x}-\vec{x}')$ type of commutation relations, when imposed on the solutions of the field equations result to a Quantum Field Theory that is

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2E_k}}\left(e^{-ik\cdot x} a_{\vec{k}} + e^{ik\cdot x} a^{+}_{\vec{k}}\right)$$

with $[a_{\vec{k}}\, a_{\vec{k}'}] = [a^{+}_{\vec{k}}, a^{+}_{\vec{k}'}] = 0$

$$[a_{\vec{k}}, a^{+}_{\vec{k}'}] = \delta(\vec{k}-\vec{k}')$$

4. But how do we get from all these the Feynman rules and the Cross-Section calculations?? ⇒ Next time

?
?
?

Common to Classical and Quantum Field Theories

Commutation relations result to Quantum Field Theories

From Free to Interacting Fields: S-Matrix

(25)

So far we considered Lagrangians that (E) did not allow for field-field interactions. But in reality the fields interact both with each other but also with themselves.

Consider for example:

$$\mathcal{L} = \underbrace{\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{m^2}{2}\phi^2}_{\text{Free-}\mathcal{L}} + \underbrace{\tfrac{\lambda}{4}\phi^4}_{\mathcal{L}_{\text{INTERACTION}}}$$

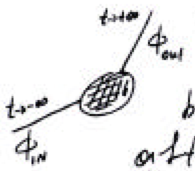
Using the Lagrange equations you get that

$$(\Box + m^2)\varphi(x) = J(x) \quad (1)$$

$$J(x) = \lambda\phi^3 \quad (2)$$

Obviously this is a complicated Diff. Equation to solve and we have to find a way out of this problem.

S-Matrix



Lets assume that the fields before they interact ($t\to-\infty$) and after they interact ($t\to+\infty$) are free ϕ_{IN}, ϕ_{out} fields respectively

(26)

So for $t\to-\infty$

$$(\Box + m^2)\phi_{IN}(x) = 0$$

$$\varphi_{IN}(x) = \int d^3k\left[a_{IN}(\vec{k})f_{\vec{k}}(x) + a^{+}_{IN}(\vec{k})f^{*}_{\vec{k}}(x)\right]$$

$$f_{\vec{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}}e^{-ikx} \quad \text{(positive freq.)}$$

and for $t\to+\infty$

$$(\Box + m^2)\phi_{out}(x) = 0$$

$$\varphi_{out}(x) = \int d^3k\left[a_{out}(\vec{k})f_{\vec{k}}(x) + a^{+}_{out}f^{*}_{\vec{k}}(x)\right]$$

As in Quantum Mechanics we always want to calculate:

$$S_{\alpha\beta} = \langle \beta\, out | \alpha_{IN}\rangle$$

$$|\alpha\ in\rangle = |P_1 P_2 \ldots P_n\ in\rangle$$

$$|\beta\ out\rangle = |P_1' P_2' \ldots P_m'\ out\rangle$$

and this because $\sigma \sim |{}_{out}\langle\beta|\alpha\rangle_{in}|^2$

(27)

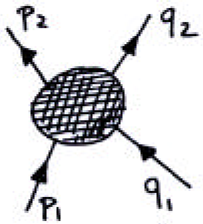
It is more formal to write:

$$\lim_{t\to+\infty} \langle\alpha|\phi(x)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\phi_{out}|\beta\rangle$$

$$\lim_{t\to-\infty} \langle\alpha|\phi(x)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\phi_{IN}|\beta\rangle$$

Consider the interaction:

Suppose $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \underbrace{\frac{\lambda}{4}\phi^4}_{\mathcal{L}_I}$



$$S = \langle p_2 q_2\, out|p_1 q_1\, IN\rangle = \langle p_2 q_2\, out|\alpha^+_{IN}(p_1)|q_1\, IN\rangle$$

$$S = \langle p_2 q_2\, out|\alpha^+_{out}(p_1)|q_1\, IN\rangle + \langle p_2 q_2\, out|\alpha^+_{IN}(p_1) - \alpha^+_{out}(p_1)|q_1\, IN\rangle$$

(Forward term — first term crossed out)

$$\boxed{S = \langle p_2 q_2\, out|\,a^+_{IN}(p_1) - a^+_{out}(p_1)\,|q_1\, IN\rangle} \quad (3)$$

Now we need a way to calculate $a^+_{IN,out}(\vec{k})$ as a function of $\Phi(x)$ ...

(28)

Recall:

$$\Phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[a_k e^{-ikx} + a^+_k e^{ikx}\right]$$

But $\int d^3x \frac{e^{ikx}}{\sqrt{(2\pi)^3 2\omega_k}}\,\phi(x) =$

$$\int d^3x \frac{e^{ikx}}{\sqrt{(2\pi)^3 2\omega_k}} \int \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}}\left[a_{k'} e^{-ik'x} + a^+_{k'} e^{ik'x}\right]$$

$$= \int \frac{d^3k'}{\sqrt{2\omega_k 2\omega_{k'}}}\frac{1}{(2\pi)^3}\int d^3x\left[a_{k'} e^{i(\vec{k}'-\vec{k})} e^{i(\omega_k-\omega_{k'})t} + a^+_{k'} e^{-i(\vec{k}'+\vec{k})} e^{i(\omega_k+\omega_{k'})t}\right]$$

$$= \int \frac{d^3k'}{\sqrt{2\omega_k 2\omega_{k'}}}\left[a_{k'}\,\delta^3(\vec{k}'-\vec{k})\, e^{i(\omega_k-\omega_{k'})t} + a^+_{k'}\,\delta^3(\vec{k}+\vec{k}')\, e^{i(\omega_k+\omega_{k'})t}\right]$$

$$= \frac{1}{2\omega_k}\left[a(\vec{k}) + a^+(-\vec{k})\, e^{2i\omega_k t}\right]$$

In Conclusion: (29)

$$\int d^3x\, f^*_{\vec{k}}(x)\, \Phi(x) = \frac{1}{2\omega_k}\left[a_{(\vec{k})} + a^\dagger_{(-\vec{k})} e^{2i\omega t}\right]$$

and (trust me)

$$\int d^3x\, f^*_{\vec{k}}(x)\, \partial_0 \phi(x) = -\frac{i}{2}\left[a_{\vec{k}} - a^\dagger_{-\vec{k}} e^{2i\omega t}\right]$$

By solving this you get

$$a(\vec{k}) = \int d^3x \left[f^*_{\vec{k}}(x)\, \omega_k \phi(x) + i f^*_{\vec{k}}(x)\, \partial_0 \phi(x)\right]$$

recall that $f^*_{\vec{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega}} e^{ikx} \rightarrow \partial_0 f^* = i\omega_k f^*$

so

$$a(\vec{k}) = \int d^3x \left[\partial_0 f^*_{\vec{k}} \left(\frac{1}{i}\right)\phi + i\, \partial_0 \phi\, f^*_{\vec{k}}(x)\right]$$

$$\boxed{\begin{aligned} a(\vec{k}) &= i \int d^3x \left(f^* \partial_0 \phi - \phi\, \partial_0 f^* \right) \\ a(\vec{k}) &= i \int d^3x\, f^*_{\vec{k}}(x)\, \overleftrightarrow{\partial_0}\, \phi(x) \end{aligned}}$$

$$f^*_{\vec{k}}(x)\, \overleftrightarrow{\partial_0}\, \phi(x) = f^*_{\vec{k}}(x)\, \partial_0 \phi(x) - \partial_0 f^*_{k}\, \phi(x)$$

Now we can go back to calculate S (30)

$$S = \langle p_2 q_2\, out | a^\dagger_{in}(p_1) - a^\dagger_{out}(p_1) | q_1\, in \rangle$$

$$S = -i \langle p_2 q_2 | \int d^3x\, f_{\vec{p}_1} \overleftrightarrow{\partial_0} \left[\phi_{in}(x) - \phi_{out}(x) \right] | q_1\, in \rangle$$

$$S = \frac{-i}{Z^{1/2}} \langle p_2 q_2 | \left(\lim_{x^0 \to -\infty} - \lim_{x^0 \to +\infty} \right) \int d^3x\, f_{\vec{p}_1} \overleftrightarrow{\partial_0} \phi(x) | q_1\, in \rangle$$

But $\left(\lim_{x^0 \to +\infty} - \lim_{x^0 \to -\infty} \right) \int d^3x\, g(x) = \int d^4x \frac{\partial g}{\partial x^0}$

$$\text{so } S = \frac{i}{Z^{1/2}} \int d^4x \frac{\partial}{\partial x^0} \left\{ f_{\vec{p}_1}(x) \overleftrightarrow{\partial_0} \phi(x) \right\} | q_1\, in \rangle$$

$$S = \frac{i}{Z^{1/2}} \int d^4x \langle p_2 q_2\, out | f_{\vec{p}_1} \frac{\partial^2 \phi(x)}{\partial x_0^2} - \frac{\partial^2 f_{p_1}(x)}{\partial x_0^2} \phi(x) | q_1\, in \rangle$$

(integrate by parts)

$$S = \frac{i}{Z^{1/2}} \int d^4x \langle p_2 q_2\, out | f_{\vec{p}_1} \frac{\partial^2 \phi(x)}{\partial x_0^2} - \phi (\vec{\nabla}^2 - m^2) f_{\vec{p}_1}(x) | q_1\, in \rangle$$

$$S = \frac{i}{\sqrt{Z}} \int d^4x_1 \langle p_2 q_2\, out | \phi(x_1) | q_1\, in \rangle \overleftarrow{(\Box_{x_1} + m^2)} f_{\vec{p}_1}(x_1)$$

$$S = \frac{i}{\sqrt{Z}} \int d^4x_1\, f_{\vec{p}_1}(x_1) \overrightarrow{(\Box_{x_1} + m^2)} \langle p_2 q_2\, out | \phi(x_1) | q_1\, in \rangle$$

The Time Ordered Product

and (31)

$$\langle p_2 q_2 \, out | p_1 q_1 \, in\rangle = \frac{i}{\sqrt{Z}} \int d^4x \, f_{p}(x) (\Box_x + m^2) \langle p_2 q_2 \, out | \phi(x) | q_1 \, in\rangle \quad (A)$$

Next we will remove the p_2 out state from ↑

$$\langle p_2 q_2 \, out | \phi(x) | q_1 \, in\rangle = \langle q_2 \, out | a_{out}(p_2) \phi(x) | q_1 \, in\rangle$$

add and sub. $\langle | \Phi a_{in}(p_2) | \rangle$

$$= \langle q_2 \, out | \Phi a_{in}(p_2) | q_1 \, in\rangle + \quad \text{(there is no } p_2 \text{ state in the input to be killed)}$$

$$\langle q_2 \, out | a_{out}(p_2) \Phi(x) - \Phi(x) a_{in}(p_2) | q_1 \, in\rangle$$

$$= \langle q_2 \, out | a_{out}(p_2) \Phi(x) - \Phi(x) a_{in}(p_2) | q_1 \, in\rangle$$

$$= +i \int d^3y \, \langle out \, q_2 | \phi_{out}(y) \Phi(x) - \Phi(x) \Phi_{in}(y) | q_1 \, in\rangle \times \overleftrightarrow{\partial}_{y_0} f^*_{\vec{p}_2}(y) \quad (B)$$

Define the time ordered product of two fields as:

$$T(a(x) b(y)) = a(x) b(y) \theta(t_x - t_y) + b(y) a(x) \theta(t_y - t_x)$$

with

$$\theta(t_i - t_j) = \begin{cases} 1 & t_i > t_j \\ 0 & t_i < t_j \end{cases}$$

(32)

Now we can take again as before the limits for
$t \to -\infty$ which will replace ϕ_{in} with ϕ and
$t \to +\infty$ which will replace ϕ_{out} with ϕ.

(B) → $\langle p_2 q_2 \, out | \Phi(x) | q_1 \, in\rangle =$

$$+\frac{i}{\sqrt{Z}} \left(\lim_{y^0 \to +\infty} - \lim_{y^0 \to -\infty} \right) \int d^3y \, \langle out \, q_2 | T \phi(y) \phi(x) | q_1 \, in\rangle \times \overleftrightarrow{\partial}_{y_0} f^*_{\vec{p}_2}(x)$$

" as you can see for $y^0 \to +\infty$ $T(\phi(y)\Phi(x)) = \Phi(y)\Phi(x)$ and
for $y^0 \to -\infty$ $T(\phi(y)\phi(x)) = \phi(x)\Phi(y)$,

As before $\langle p_2 q_2 \, out | \Phi(x) | q_1 \, in\rangle =$

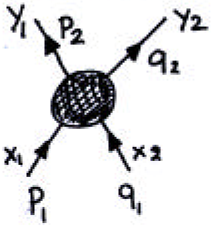
$$\frac{i}{\sqrt{Z}} \int d^4x \, \frac{\partial}{\partial x^0} \left\{ \langle out \, q_2 | T(\phi(y)\phi(x)) | q_1 \, in\rangle \times \overleftrightarrow{\partial}_{y_0} f^*_{\vec{p}_2}(x) \right\} \quad (C)$$

Finally using (A) (C) we get that

$$\langle p q_2 \, out | p_1 q_1 \, in\rangle = \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4y \int d^4x \, f_{\vec{p}_1}(x) f^*_{2}(y) (\Box_x + m^2)(\Box_y + m^2) \langle q_2 \, out | T \phi(y) \phi(x) | q_1 \, in\rangle$$

Summary

SUMMARY: We want to calculate the amplitude (33) (S-matrix) for the reaction $p_1+q_1 \to p_2+q_2$ where p_1, p_2, q_1, q_2 are spin zero (scalar) particles described by a theory with



$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

We have found that: $\langle p_2 q_2\ \mathrm{out} | p_1 q_1\ \mathrm{in}\rangle =$

$$\left(\frac{i}{\sqrt{z}}\right)^4 \int d^4y_1 \int d^4y_2 \int d^4x_1 \int d^4x_2\, f_{\vec{p}_1}(x_1)\, f_{\vec{q}_1}(x_2)\, f^*_{\vec{p}_2}(y_1)\, f^*_{\vec{q}_2}(y_2)$$

$$\times\ (\Box_{x_1}+m^2)(\Box_{x_2}+m^2)(\Box_{y_1}+m^2)(\Box_{y_2}+m^2).$$

$$\langle 0|T\,\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)|0\rangle$$

By making the $\Box$ operators act on $f(x)$'s (by integrating twice by parts and dropping the surface terms) we get: $\langle p_2 q_2\ \mathrm{out}|p_1 q_1\ \mathrm{in}\rangle =$

$$= \left(\frac{i}{\sqrt{z}}\right)^4 \int d^4y_1 \int d^4y_2 \int d^4x_1 \int d^4x_2\ e^{-i(q_1 x_2 + p_1 x_1 - q_2 y_2 - p_2 y_1)}$$

$$\times \frac{(q_1^2-m^2)}{\sqrt{(2\pi)^3 2E_{q_1}}} \times \frac{(q_2^2-m^2)}{\sqrt{(2\pi)^3 2E_{q_2}}} \times \frac{(p_1^2-m^2)}{\sqrt{(2\pi)^3 2E_{p_1}}} \times \frac{(p_2^2-m^2)}{\sqrt{(2\pi)^3 2E_{p_2}}}$$

$$\langle 0|T(\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2))|0\rangle$$

Wick's Theorem

Normal Ordering and Contracted Product of Operators. WICKS THEOREM: (34)

In order to calculate $\langle 0|T(\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2))|0\rangle$ we need to know a few more "tricks"....

Recall that $\phi(x) = \int d^3k \left[a_k f_k(x) + a_k^+ f_k^*(x) \right]$ with

$$f_k(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ikx} \text{ (positive freq. part)}$$

① DEFINE THE N OPERATOR WHICH ORDERS ANNIHILATION OPERATORS (positive frequency parts) to THE RIGHT OF THE CREATION OPERATORS. THIS IS CALLED "NORMAL ORDERING" OR "NORMAL PRODUCT". It works like this:

$$N(\phi(x)\phi(y)) = N\left((\phi^+(x)+\phi^{(-)}(x))(\phi^+(y)+\phi^{(-)}(y))\right)$$

(positive freq. ; neg. freq.)

$$= N\left(\phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{+}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y)\right)$$

$$= \phi^+(x)\phi^+(y) + \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(y)\phi^{(+)}(x) + \phi^{(-)}(x)\phi^{(+)}(y)$$

Some times $N(\phi(x)\phi(y))$ is written as $:\phi(x)\phi(y):$ and it is one and the same thing.

(35)

Define the contracted field Product by:

$$\varphi^\circ(x)\varphi^\circ(y) \equiv T(\phi(x)\phi(y)) - N(\phi(x)\phi(y))$$

Lets try to calculate $\varphi^\circ(x)\varphi^\circ(y)$ for $x^0 > y^0$

$$\varphi^\circ(x)\varphi^\circ(y) = T\left\{(\Phi^{(+)}(x)+\Phi^{(-)}(x))\cdot(\Phi^{(+)}(y)+\Phi^{(-)}(y))\right\} + \\ - N\left\{(\Phi^{(+)}(x)+\Phi^{(-)}(x))\cdot(\Phi^{(+)}(y)+\Phi^{(-)}(y))\right\} \Rightarrow$$

$$\varphi^\circ(x)\varphi^\circ(y) = \Phi^{(+)}(x)\Phi^{(+)}(y) + \Phi^{(+)}(x)\Phi^{(-)}(y) + \Phi^{(-)}(x)\Phi^{(+)}(y) + \Phi^{(-)}(x)\Phi^{(-)}(y) + 0 + \\ - \left(\Phi^{+}(x)\Phi^{(+)}(y) + \Phi^{(-)}(x)\Phi^{(-)}(y) + \Phi^{(-)}(y)\Phi^{(+)}(x) + \Phi^{(-)}(x)\Phi^{(+)}(y)\right) \Rightarrow$$

($y^0 > x^0$ contrib. to T())

$$\varphi^\circ(x)\varphi^\circ(y) = \Phi^{(+)}(x)\Phi^{(-)}(y) - \Phi^{(-)}(y)\Phi^{(+)}(x) \Rightarrow$$

$$\boxed{\Phi^\circ(x)\Phi^\circ(y) = \left[\Phi^{(+)}(x), \Phi^{(-)}(y)\right]}$$

(36)

So

$$\Phi^{\circ}(x)\Phi^{\circ}(y)=[\Phi^{(+)}(x),\Phi^{(-)}(y)]=\frac{1}{(2\pi)^3}\int\frac{d^3k}{\sqrt{2E_k}}\int\frac{d^3k'}{\sqrt{2E_{k'}}}e^{-ikx}e^{ik'y}[a_{\vec{k}},a^{\dagger}_{\vec{k}'}]$$

Since $\Phi(x)$ is a quantum field we have that

$$[a_{\vec{k}},a^{\dagger}_{\vec{k}'}]=\delta^{(3)}(\vec{k}-\vec{k}')$$

Therefore:

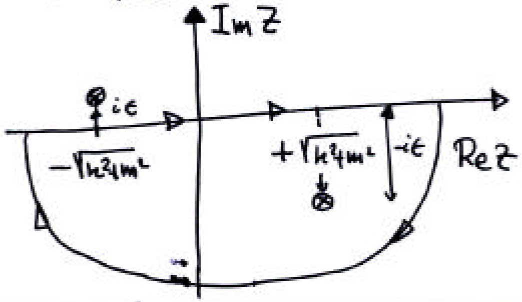
$$\Phi^{\circ}(x)\Phi^{\circ}(y)=\frac{1}{(2\pi)^3}\int\frac{d^3k}{\sqrt{2E_k}}\int\frac{d^3k}{\sqrt{2E_{k'}}}\delta^3(k-k')e^{-ik\cdot x+ik'y}$$

and $$\Phi^{\circ}(x)\Phi^{\circ}(y)=[\Phi^{(+)}(x),\Phi^{(-)}(y)]=\frac{1}{(2\pi)^3}\int\frac{d^3k}{2E_k}e^{-ik\cdot(x-y)} \quad (A)$$

But $$D_F(x)=\frac{i}{(2\pi)^4}\int d^4k\frac{e^{-ikx}}{k^2-m^2}=\frac{i}{(2\pi)^4}\int d^3k\int dk^0\frac{e^{-ikx}}{k^2-m^2}\Rightarrow$$

$$D_F(x)=\frac{i}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\underbrace{\int_{-\infty}^{+\infty}dk^0\frac{e^{-ik^0x^0}}{(k^0-\sqrt{k^2+m^2})(k^0+\sqrt{k^2+m^2})}}_{I} \quad (C)$$

Recall from Complex analysis that I can be evaluated as follows



(37)

$$I=\int_{-\infty}^{+\infty}\frac{dk^0\, e^{-ik^0x^0}}{(k^0-\sqrt{k^2+m^2}+i\epsilon)(k^0+\sqrt{k^2+m^2}-i\epsilon)}$$

$$I=-2\pi i\,\mathrm{Res}\left\{\frac{e^{-ik^0x^0}}{(k^0-\underbrace{\sqrt{k^2+m^2}}_{\omega_k}+i\epsilon)(k^0+\underbrace{\sqrt{k^2+m^2}}_{\omega_k}-i\epsilon)}\right\}$$

$$I=-2\pi i\left\{\frac{e^{-ik^0x^0}\,(k^0-\omega_k+i\epsilon)}{(k^0-\omega_k+i\epsilon)(k^0+\omega_k-i\epsilon)}\right\}_{k^0=\omega_k}$$

$$I=-2\pi i\,\frac{e^{-i\omega_k x^0}}{2\omega_k} \quad (B)$$

$$(B)+(C)\rightarrow D_F(x)=\frac{i}{(2\pi)^4}\int d^3\vec{k}\,(-2\pi i)\,e^{i\vec{k}\cdot\vec{x}}\frac{e^{-i\omega_k x^0}}{2\omega_k}\Rightarrow$$

$$D_F(x)=\frac{1}{(2\pi)^3}\int\frac{d^3\vec{k}}{2\omega_k}e^{-ikx} \quad \text{or} \quad D_F=\frac{1}{(2\pi)^3}\int\frac{d^3k}{2E_k}e^{-ikx}$$

In Conclusion:

$$\Phi^{\circ}(x)\Phi^{\circ}(y)=[\Phi^{(+)}(x),\Phi^{(-)}(y)]=\frac{1}{(2\pi)^4}\int d^4k\, e^{-ikx}\times\frac{i}{k^2-m^2}$$

$\frac{i}{k^2-m^2}$ is the scalar field propagator!!

Summary

(38)

If one wonders by now why are all these usefull here is the answer:

Our goal was to evaluate the Vaccum expectation Value of the time ordered product: $\langle 0|T\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)|0\rangle$

The contructed operator formula gives us almost what we want:

$$\langle 0|T\Phi(x)\Phi(y)|0\rangle = \underbrace{\langle 0|\overset{\circ}{\Phi}(x)\overset{\circ}{\Phi}(y)|0\rangle}_{\text{we know by now what is this and how is it calculated}} +$$

$$+\underbrace{\langle 0|N(\Phi(x)\Phi(y))|0\rangle}_{\to 0} \quad \text{since } a_k|0\rangle = 0$$

We only need to make this more general So it applies to 4-fields. To do this we use the theorem discovered by G.C Wick

Wick's Theorem

(39)

G.C. Wick, Phys. Rev. 80, p268, 1950

$$T(\Phi(x_1)\Phi(x_2)\ldots\Phi(x_n)) =$$

$$N(\Phi(x_1)\ldots\Phi(x_n)) +$$

$$+\sum_{i<j}\overset{\circ}{\Phi}(x_i)\overset{\circ}{\Phi}(x_j)\,N(\Phi(x_a)\ldots\Phi(x_b))_{a\ldots b\neq i,j} +$$

$$+\sum_{\substack{i<j\\k<l}}\overset{\circ}{\Phi}(x_i)\overset{\circ}{\Phi}(x_j)\overset{\circ\circ}{\Phi}(x_k)\overset{\circ\circ}{\Phi}(x_l)\,N(\Phi(x_a)\ldots\Phi(x_b))_{\substack{a\ldots b\neq i,j\\a\ldots b\neq k,l}}$$

$$+\ \ldots\ldots$$

Examples

Examples:

1. $$T(\phi) = \phi$$

2. $$T(\phi(x)\phi(y)) = \dot{\phi}(x)\dot{\phi}(y) + N(\phi(x)\phi(y))$$

3. $$T(\phi(x)\phi(y)\phi(z)) = N(\phi(x)\phi(y)\phi(z)) + \dot{\phi}(x)\dot{\phi}(y)\,\phi(z) + \dot{\phi}(x)\dot{\phi}(z)\,\phi(y) + \dot{\phi}(y)\dot{\phi}(z)\,\phi(x)$$

4. $$T(\phi_1\phi_2\phi_3\phi_4) = N(\phi_1\phi_2\phi_3\phi_4) + \dot{\phi}_1\dot{\phi}_2 N(\phi_3\phi_4) + \dot{\phi}_1\dot{\phi}_3 N(\phi_2\phi_4) + \dot{\phi}_1\dot{\phi}_4 N(\phi_2\phi_3) + \dot{\phi}_2\dot{\phi}_3 N(\phi_1\phi_4) + \dot{\phi}_2\dot{\phi}_4 N(\phi_1\phi_3) + \dot{\phi}_3\dot{\phi}_4 N(\phi_1\phi_2) + \dot{\phi}_1\dot{\phi}_2\ddot{\phi}_3\ddot{\phi}_4 + \dot{\phi}_1\dot{\phi}_3\ddot{\phi}_2\ddot{\phi}_4 + \dot{\phi}_1\dot{\phi}_4\ddot{\phi}_2\ddot{\phi}_3 +$$

$\phi_i = \phi(x_i)$

Perturbation Theory

Our goal is to evaluate the time-ordered (41) product of fields:

$$\langle 0 | T\, \Phi(x_1)\Phi(y_1)\Phi(x_2)\Phi(y_2) | 0\rangle$$

The problem is that we do not know the $\Phi(x_i)$ fields and we would like to express them in terms of the $\Phi_{IN}(x_i)$ fields which we know from the free-field equations. Therefore we define the unitary operator U such that:

$$\left.\begin{aligned} \Phi_{IN}(x) &= U\,\Phi(x)\,U^{-1} \\ \Phi(x) &= U^{-1}\,\Phi_{IN}(x)\,U \end{aligned}\right\} \quad (1)$$

In the Heisenberg picture we have that

$$\dot{\Phi}_{IN} = i\left[H_{IN}(\Phi_{IN},\Pi_{IN}),\Phi_{IN}\right] \quad (2)$$

$$\dot{\Phi} = i\left[H(\phi,\pi),\Phi\right] \quad (3)$$

$(1) \rightarrow \dot{\Phi}_{IN} = \dot{U}(t)\,\Phi\,U^{-1}(t) + U\dot{\Phi}U^{-1} + U\Phi\dot{U}^{-1} \Rightarrow$

$\dot{\Phi}_{IN} = \dot{U}U^{-1}\Phi_{IN}UU^{-1} + U\dot{\Phi}U^{-1} + UU^{-1}\Phi_{IN}U\dot{U}^{-1} \rightarrow$

$$\left.\begin{aligned} &\dot{\Phi}_{IN} = \dot{U}U^{-1}\Phi_{IN} + \Phi_{IN}U\dot{U}^{-1} + U\dot{\Phi}U^{-1} \\ &\text{but}\quad UU^{-1} = 1 \Rightarrow \dot{U}U^{-1} = -U\dot{U}^{-1} \end{aligned}\right\} \Rightarrow$$

(42)

$\dot{\Phi}_{IN} = \dot{U}U^{-1}\Phi_{IN} - \Phi_{IN}\dot{U}U^{-1} + U\dot{\Phi}U^{-1} \rightarrow$

$\dot{\Phi}_{IN} = [\dot{U}U^{-1},\Phi_{IN}] + U\dot{\Phi}U^{-1} \overset{(3)}{\rightarrow}$

$\dot{\Phi}_{IN} = [\dot{U}U^{-1},\Phi_{IN}] + U\{i[H(\phi,\pi),\Phi]\}U^{-1} \rightarrow$

$\dot{\Phi}_{IN} = [\dot{U}U^{-1},\Phi_{IN}] + i[H(\phi_{IN},\Pi_{IN}),\Phi_{IN}]$

Define $H(\phi_{IN}\,\Pi_{IN}) = H_{IN}(\phi_{IN},\Pi_{IN}) + H_I(\phi_{IN}\Pi_{IN})$
$\qquad = H_{FREE}(\phi_{IN},\Pi_{IN}) + H_I(\phi_{IN},\Pi_{IN})$ $\Rightarrow$

$$\cancel{\dot{\Phi}_{IN}} = [\dot{U}U^{-1},\Phi_{IN}] + \underbrace{i[H_{IN}(\phi_{IN},\Pi_{IN}),\Phi_{IN}]}_{\cancel{\dot{\Phi}_{IN}}} + i[H_I(\phi_{IN},\Pi_{IN}),\phi_{IN}]$$

$$\therefore\ [\dot{U}U^{-1} + iH_I(\phi_{IN},\Pi_{IN}),\ \Phi_{IN}] = 0$$

$$\therefore\ -i\dot{U}U^{-1} + H_I(\Phi_{IN},\Pi_{IN}) = 0$$

$$\boxed{i\frac{\partial U}{\partial t} = H_I(\phi_{IN},\Pi_{IN})\cdot U(t)}$$

This equation can be solved as follows: (43)

$$\frac{\partial \mathcal{U}}{\partial t} = -i\, H_I(t)\, \mathcal{U}(t) \Rightarrow$$

$$\left.\int_{t_1}^{t_2} \frac{\partial \mathcal{U}}{\partial t} dt = -i \int_{t_1}^{t_2} H_I(t)\, \mathcal{U}(t)\, dt \right\} \Rightarrow$$

set $\mathcal{U}(t_1) = 1$

$$\mathcal{U}(t_2) = 1 - i \int_{t_1}^{t_2} H_I(t)\, \mathcal{U}(t)\, dt$$

$$t_1 \to -\infty,\ t_2 = t \Rightarrow \mathcal{U}(t) = 1 - i \int_{-\infty}^{t} H_I(t)\, \mathcal{U}(t)\, dt$$

$$\mathcal{U}(t) = 1 - i \int_{-\infty}^{t} H_I(t')\, dt' + (-i)^2 \int_{-\infty}^{t} H_I(t') \int_{-\infty}^{t'} H_I(t'')\, dt' dt'' + \ldots \quad t > t' > t''$$

At the end $\mathcal{U}(t) = T\, e^{-i \int_{-\infty}^{t} H_I(t)\, dt}$ or

$$\mathcal{U}(t) = T\, e^{-i \int d^4x\, \mathcal{H}_I}$$

(where H_I is the Hamiltonian and $\mathcal{H}$ is the Hamiltonian density)

Now we can go back to the time ordered product (44)

$$\langle 0 | T \{ \Phi(x_1) \Phi(y_1) \Phi(x_2) \Phi(y_2) | 0 \rangle =$$

$$\langle 0 | T \{ \mathcal{U}^{-1}(x_1^0)\, \Phi_{IN}\, \mathcal{U}(x_1^0)\, \mathcal{U}^{-1}(y_1^0)\, \Phi_{IN}(y_1)\, \mathcal{U}(y_1^0)$$
$$\mathcal{U}^{-1}(x_2^0)\, \Phi_{IN}(x_2)\, \mathcal{U}(x_2^0)\, \mathcal{U}^{-1}(y_2^0)\, \Phi_{IN}(y_2)\, \mathcal{U}(y_2) \} | 0 \rangle$$

Since this is a time ordered product, inside the product we can combine terms as we want. Assume for example $x_1^0 > y_1^0 > x_2^0 > y_2^0$.

At the end we have that:

$$\langle 0 | T (\Phi(x_1) \Phi(y_1) \Phi(x_2) \Phi(y_2) | 0 \rangle =$$
$$\langle 0 | T \{ \Phi_{IN}(x_1) \Phi_{IN}(x_2) \Phi_{IN}(y_1) \Phi_{IN}(y_2)\, e^{-i \int d^4x\, \mathcal{H}_I} \} | 0 \rangle$$

So if we have the Φ_{IN} fields that we get from the free field equations and the interaction part $\mathcal{H}_I$ of the Hamiltonian then we can calculate the amplitude for the process.

Interaction Lagrangian and Disconnected Diagrams

(45)

Recall that $\mathcal{L} = \mathcal{L}_{FREE} - \frac{\lambda}{4!}\Phi^4 \Rightarrow$

$$\mathcal{H}_I(\phi) = -\frac{\lambda}{4!}\Phi^4$$

$$\langle 0|T\left[\Phi_{IN}(x_1)\Phi_{IN}(y_1)\Phi_{IN}(x_2)\Phi_{IN}(y_2)\, e^{-i\int d^4x\, \mathcal{H}_I(x)}\right]|0\rangle =$$

$$= \langle 0|T\left[\Phi_{IN}(x_1)\Phi_{IN}(y_1)\Phi_{IN}(x_2)\Phi_{IN}(y_2)\left(1 - i\int d^4x\, \mathcal{H}(x) + \frac{(-i)}{2!}\int d^4x \int d^4y\, \mathcal{H}(x)\mathcal{H}(y) \dots\right)\right]|0\rangle$$

The terms of order λ^0 are

$$G^{(0)} = \langle 0|T\left(\phi_{IN}(x_1)\Phi_{IN}(x_2)\Phi_{IN}(y_1)\phi_{IN}(y_2)\right)|0\rangle$$

(WICK'S THEOREM)

$$= \langle 0|N\left(\phi_{IN}(x_1)\dots\phi_{IN}(y_2)\right)|0\rangle + \quad (\to 0)$$

$$\langle 0|\ \Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(x_2)\, N(\Phi_{IN}(y_1)\Phi_{IN}(y_2)) + \dots \dots \quad (\to 0)$$

many terms with $N(\ )$

$$\Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(y_1)\Phi^{\circ\circ}_{IN}(x_2)\Phi^{\circ\circ}_{IN}(y_2) +$$
$$\Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(x_2)\Phi^{\circ\circ}(y_1)\Phi^{\circ\circ}(y_2) +$$
$$\Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(y_2)\Phi^{\circ\circ}(x_2)\Phi^{\circ\circ}(y_1)\ |0\rangle$$

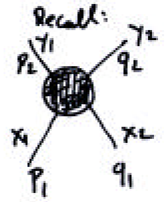
Non vanishing yet

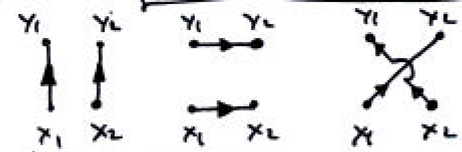
(46)

Recall that the contracted operator product for scalar fields was:

$$\varphi^{\circ}(x)\varphi^{\circ}(y) = D_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k \frac{i}{k^2-m^2} e^{-ik(x-y)}$$

Then: $G^{(0)} = D_F(x_1-y_1)D_F(x_2-y_2) + D_F(x_1-x_2)D_F(y_1-y_2) + D_F(x_1-y_2)D_F(x_2-y_1)$





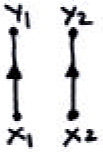
The D_F products can be written as diagrams

so the contribution of these diagrams to the amplitude is:

$$\langle p_2 q_2\, out|p_1 q_1\, in\rangle = \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \times$$

$$\times \frac{q_1^2 - m^2}{\sqrt{(2\pi)^3 2E_{q_1}}} \times \frac{q_2^2-m^2}{\sqrt{(2\pi)^3 2E_{q_2}}} \times \frac{p_1^2-m^2}{\sqrt{(2\pi)^3 2E_{p_1}}} \times \frac{p_2^2-m^2}{\sqrt{(2\pi)^3 2E_{p_2}}}\, e^{-i(q_1x_2 + p_1x_1 - p_2y_1 - q_2y_2)} \times$$

$$\times \left\{D_F(x_1-y_1)D_F(x_2-y_2) + D_F(x_1-x_2)D_F(y_1-y_2) + D_F(x_1-y_2)D_F(x_2-y_1)\right\}$$

Lets Calculate the  contribution: (47)

$$T_1^{(0)} = \left(\frac{i}{\sqrt{2}}\right)^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \times \frac{q_1^2-m^2}{\sqrt{2E_{q_1}(2\pi)^3}} \times \frac{q_2^2-m^2}{\sqrt{2E_{q_2}(2\pi)^3}} \times \frac{p_1^2-m^2}{\sqrt{2E_{p_1}(2\pi)^3}} \times \frac{p_2^2-m^2}{\sqrt{2E_{p_2}(2\pi)^3}} \times$$

$$\times e^{-i(q_1x_2+p_1x_1-p_2y_1-q_2y_2)} \times \frac{1}{(2\pi)^4}\int d^4k_1 \frac{i}{k_1^2-m^2} e^{-ik_1(x_1-y_1)} \times$$

$$\times \frac{1}{(2\pi)^4}\int d^4k_2 \frac{i}{k_2^2-m^2} e^{-ik_2(x_2-y_2)} \Rightarrow$$

$$T_1^{(0)} = \left(\frac{i}{\sqrt{2}}\right)^4 \frac{q_1^2-m^2}{\sqrt{2E_{q_1}(2\pi)^3}} \times \frac{q_2^2-m^2}{\sqrt{2E_{q_2}(2\pi)^3}} \times \frac{p_1^2-m^2}{\sqrt{2E_{p_1}(2\pi)^3}} \times \frac{p_2^2-m^2}{\sqrt{2E_{p_2}(2\pi)^3}} \times \left(\frac{1}{(2\pi)^4}\right)^2 \times$$

$$\times \int d^4k_1 \frac{i}{k_1^2-m^2} \int d^4k_2 \frac{i}{k_2^2-m^2} \times \underbrace{\int d^4x_1\, e^{-i(k_1+p_1)\cdot x_1}}_{(2\pi)^4\delta^{(4)}(k_1+p_1)} \times$$

$$\underbrace{\int d^4x_2\, e^{-i(k_2+q_1)x_2}}_{(2\pi)^4\delta^4(k_2+q_1)} \int d^4y_1\, e^{+i(p_2+k_1)y_1} \int d^4y_2\, e^{i(q_2+k_2)\dots}$$

$$\Rightarrow T_1^{(0)} = \left(\frac{i}{\sqrt{2}}\right)^4 \left(\frac{1}{(2\pi)^4}\right)^2 \left((2\pi)^4\right)^2 \frac{q_1^2-m^2}{\sqrt{\ }} \times \frac{q_2^2-m^2}{\sqrt{\ }} \times \frac{p_1^2-m^2}{\sqrt{\ }} \times \frac{p_2^2-m^2}{\sqrt{\ }} \times$$

$$\times \frac{i}{p_1^2-m^2} \times \frac{i}{q_1^2-m^2} \underbrace{\int d^4y_1\, e^{i(p_2-p_1)}}_{(2\pi)^4\delta^4(p_2-p_1)} \underbrace{\int d^4y_2\, e^{i(q_2-q_1)}}_{(2\pi)^4\delta(q_1-q_2)}$$

(48)

$$T_1^{(0)} = \left(\frac{i}{\sqrt{2}}\right)^4 \left((2\pi)^4\right)^2 \delta^{(4)}(p_2-p_1)\, \delta^{(4)}(q_2-q_1) \times$$

$$\times\, i^2 \frac{q_2^2-m^2}{\sqrt{2E_{q_2}(2\pi)^3}} \times \frac{p_2^2-m^2}{\sqrt{2E_{p_2}-m^2}}$$

Since $q_2, p_2 \to m^2$ (real particles at initial + final states)

We see that $T_1^{(0)} = 0$

In fact for the same reason the other two diagrams vanish (not enough q^2-m^2 factors in the denominator). And $G^{(0)} = 0$

Order ? Connected Diagram

(49)

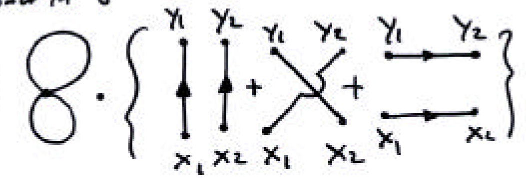
Next calculate the term which is of order λ^1.

$$G^{(1)} = -\frac{i}{4!}\lambda \int d^4x \langle 0| T\,\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Phi^4(x)|0\rangle$$

using Wick's theorem, the IN terms vanish and we get:

$$T(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Phi^4(x)) =$$

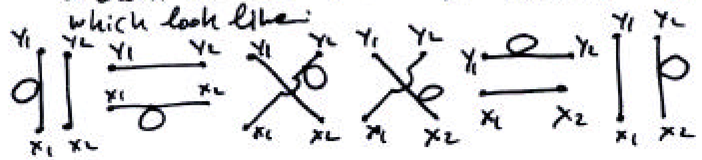
$$\underbrace{3\,\Phi^{\circ}(x)\Phi^{\circ}(x)\Phi^{\circ\circ}(x)\Phi^{\circ\circ}(x)}_{\text{3 ways of doing this as we saw in } G^{(0)}} \cdot \underbrace{G^{(0)}(x_1 x_2 y_1 y_2)}_{\text{as before}} + \text{more terms written below}$$

$= 3$ 

$$+\binom{4}{2}\varphi^{\circ}(x)\varphi^{\circ}(x)\left[2\Phi^{\circ\circ}(x)\Phi^{\circ\circ}(x_1)\Phi^{\circ\circ\circ}(x)\Phi^{\circ\circ\circ}(x_2)\right]\Phi^{\circ\circ\circ\circ}(y_1)\Phi^{\circ\circ\circ\circ}(y_2)$$

$+ \ldots$

a class of terms of the type $\varphi^{\circ}(x)\varphi^{\circ}(x)[\quad]\Phi(y_1)\Phi(y_2)$ which look like:

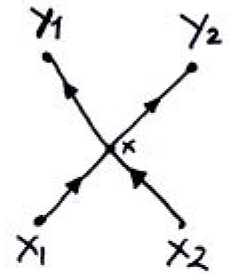


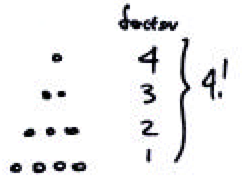
(50)

The previous two classes of diagramms contain disconnected diagramms and as we already know vanish (not enough denominators). But there is also one more order λ graph which is:

$$4!\,\phi^{\circ}(x)\phi^{\circ}(x_1)\phi^{\circ\circ}(x)\phi^{\circ\circ}(y_1)\phi^{\circ\circ\circ}(x)\phi^{\circ\circ\circ}(y_2)\Phi^{\circ\circ\circ\circ}(x)\phi^{\circ\circ\circ\circ}(x_2)$$

which can be written as





and does have enough $\frac{1}{k^2 m^2}$ terms ...

This term can be evaluated as follows.

Result and Higher Order Terms

(51)

$$G^{(4)} = \left(\frac{i}{\sqrt{Z}}\right)^4 \iint d^4y_1 \int d^4y_2 \int d^4x_1 \int d^4x_2 \, e^{-i(q_1x_2 + P_1x_1 - q_2y_2 - P_2y_1)} \, d^4x \times$$

this is from $\int d^4 y_H$

$$\times \frac{P_1^2-m^2}{\sqrt{2E_{p_1}}(2\pi)^3} \times \frac{P_2^2-m^2}{\sqrt{2E_{p_2}}(2\pi)^3} \times \frac{q_1^2-m^2}{\sqrt{2E_{q_1}}(2\pi)^3} \times \frac{q_2^2-m^2}{\sqrt{2E_{q_2}}(2\pi)^3} \times \int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4$$

$$4! \times \left(\frac{i}{(2\pi)^4}\right)^4 \underbrace{\frac{e^{iK_1(x_1-x)}}{k_1^2-m^2} \times \frac{e^{ik_2(x_2-x)}}{k_2^2-m^2} \times \frac{e^{ik_3(y_1-x)}}{k_3^2-m^2} \times \frac{e^{ik_4(y_2-x)}}{k_4^2-m^2}}_{\text{PROPAGATORS OF } s=0 \text{ FIELDS}} \times$$

$$\times \frac{-i}{4!} \times \lambda$$

① The $e^{ix_i \cdot k_i}$ terms will pair with the $e^{iq_i x_i}$ terms to produce $\delta^4(\;)\,(2\pi)^4$ terms (4 of them) and each time they do it they will "spend" a $\int d^4x_i$.

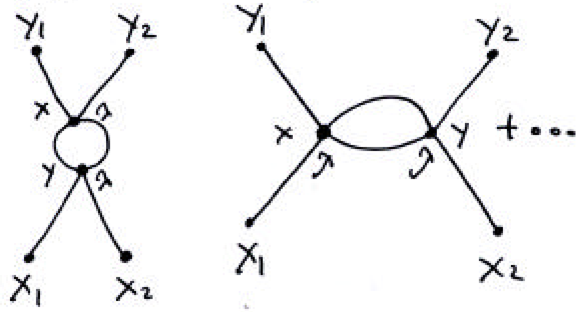
② the $\delta^4()$ functions then will kill one $\int d^4k_i$ term leaving at the end $\int d^4x \, e^{-i(k_1+k_2-k_3-k_4)x}$

At the end we have:

$$\boxed{\langle P_2 q_2 \, \text{out} | P_1 q_1 \, \text{in} \rangle = \frac{-i\lambda}{2!} \frac{(2\pi)^4 \, \delta(q_1+P_1-q_2-P_2)}{\sqrt{2E_{q_1}(2\pi)^3}\sqrt{2E_{q_1}(2\pi)^3}\sqrt{2E_{q_1}(2\pi)^3}\sqrt{2E_{p_2}(2\pi)^3}}}$$

(52)

At λ^2-order things will look like



which may be written as

$$\frac{(-i)^2}{2!}\left(\frac{\lambda}{4!}\right)\int d^4x \int d^4y \times$$

$$\langle 0| T(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Phi^4(x)\Phi^4(y))|0\rangle$$

Clearly we need some rules to calculate these diagrams so we don't have to repeat all these calculations every time we want to evaluate the cross section of a process.

Global and Local Abelian Symmetries

GLOBAL AND LOCAL SYMMETRIES OF THE LAGRANGIAN (53)

Consider the transformation where:

$$\Phi \to e^{ie}\Phi \quad \text{where } e \text{ is constant.}$$

The Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - \frac{m^2}{2}\Phi^*\Phi$ is invariant under this transformation and using Noether's theorem we have that

$$J^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\delta\Phi_i$$

(Φ, Φ^* are two independent degrees of freedom. If you prefer $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$)

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i^*}\delta\Phi_i^* + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\delta\Phi_i \Rightarrow$$

$$\delta\Phi = ie\Phi$$

$$J^\mu = \frac{1}{2}\partial^\mu\Phi(-ie\Phi^*) + \frac{1}{2}\partial^\mu\Phi^*(ie\Phi) \to$$

$$J^\mu = \frac{ie}{2}\left(\Phi\partial^\mu\Phi^* - \Phi^*\partial^\mu\Phi\right)$$

and by Noether's theorem $\partial_\mu J^\mu = 0$

∴ An internal global symmetry (continuous) leads to a conserved current.

(54)

This kind of transformations are called global transformations because the transformation has e^{ie} where e is a constant which is the same everywhere in space-time. One could have a constant matrix then which would be the generator of an SU(n) group.

$e^{ie} \to U(1)$ group.

One can extend this concept by letting the exponent to be space time dependent. Then we have the local transformations:

Local U(1): $\Phi(x) \to \Phi'(x) = e^{ie\alpha(x)}\Phi(x)$

(note that $\alpha(x)$ is space-time dep.)

U(1) abelian group:

1. $1 = e^{i0}$
2. $\forall\, g_i = e^{i\theta_i} \Rightarrow \exists\, e^{-i\theta_i} = g^{-1} : g \cdot g^{-1} = g^{-1} g = 1$
3. $g_i g_j = e^{i\theta_i} e^{i\theta_j} = e^{i(\theta_i+\theta_j)} = g_j g_i$
4. if $g_i \cdot g_j = g_e \Rightarrow g_e$ belongs to the U(1) group

Local Symmetries

(55)

Demand that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\nu\phi^*\partial^\mu\phi - \frac{m^2}{2}\Phi^*\Phi$$

is invariant under $\Phi \to e^{i\alpha(x)\cdot e}\Phi(x) = \Phi'(x)$

That is: The Lagrangian density is invariant under U(1) LOCAL GAUGE TRANSFORMATIONS. (We will see later why "GAUGE").

$$\Phi^*\Phi \to \Phi^{*\prime}\Phi' = e^{-ie\alpha(x)}\Phi^* e^{ie\alpha(x)}\Phi = \Phi^*\Phi$$

$\Rightarrow$ INVARIANT

$$\partial_\mu\Phi \to \partial_\mu\Phi' = \partial_\mu\left\{e^{i\alpha(x)\cdot e}\Phi(x)\right\}$$

$$\to e^{i\alpha(x)\cdot e}\partial_\mu\Phi(x) + ie\,\partial_\mu\alpha(x)e^{i\alpha(x)e}\Phi(x)$$

$$\to \underbrace{e^{i\alpha(x)e}\partial_\mu\Phi(x)}_{\text{ok term the phase will go away when } \partial_\mu\phi^*\partial^\mu\phi} + \underbrace{e^{i\alpha(x)e}\Phi(x)(-ie\partial_\mu\alpha(x))}_{\text{No good, depends upon } \alpha(x) \text{ and this will not go away...}}$$

Clearly if we want $\mathcal{L}$ invariant we must change something to fix this problem.

(56)

The only reasonable option we have is to change $\partial_\mu \to D_\mu$ (yet to be defined) which somehow will kill the $\partial_\mu\alpha(x)$ terms.

Introduce a vector field A_μ (∂_μ is a vector) such that $D_\mu = \partial_\mu - ieA_\mu(x)$

The idea is that while $\Phi \to \Phi'$ $A_\mu(x)$ also $A_\mu \to A_\mu'$ such that $\mathcal{L}$ remains invariant.

In math:

$$(\partial_\mu - ieA_\mu')\underbrace{U\Phi(x)}_{\Phi'(x)} = U(\partial_\mu - ieA_\mu)\Phi(x)$$

$$\partial_\mu(e^{ie\alpha(x)}\Phi(x)) - ieA_\mu' e^{ie\alpha(x)}\Phi(x) =$$

$$e^{ie\alpha(x)}\partial_\mu\Phi(x) - ieA_\mu\Phi(x)e^{ie\alpha(x)} \to$$

$$ie\,\partial_\mu\alpha(x)\Phi(x)e^{ie\alpha(x)} + e^{ie\alpha(x)}\partial_\mu\Phi(x) - ieA_\mu' e^{ie\alpha(x)}\Phi(x) = e^{ie\alpha(x)}\partial_\mu\Phi(x) - ieA_\mu\Phi(x)e^{ie\alpha(x)} \to$$

$$ie\,\partial_\mu\alpha(x) - ieA_\mu' = -ieA_\mu \to$$

$$\boxed{A_\mu' = A_\mu + \partial_\mu\alpha(x)}$$

Summary and Conclusions

(57)

Summary: To make $\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2}{2}\phi^*\phi$ invariant under U(1) where

$$\Phi \to \Phi'(x) = e^{ie\alpha(x)}\Phi(x)$$

we introduce

① $D_\mu = \partial_\mu - ieA_\mu$, D_μ is called COVARIANT DERIVATIVE and A_μ is a spin 1 (Vector) field

② $A'_\mu = A_\mu + \partial_\mu\alpha(x)$ (gauge)

One can immediately see that:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

is invariant under local U(1) transformations

It is also obvious that a term

$$m^2 A_\mu A^\mu$$

which would give mass to the field A^μ is prohibited by local gauge invariance

(58)

In conclusion the full Lagrangian of this theory can be written as:

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{m^2}{2}\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This theory has the U(1) local gauge symmetry. The "gauge" term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ gives us the Maxwell Eq. Therefore it describes the vector field A_μ called photon.

The first prediction of our U(1) theory is that the photon mass $m_\gamma = 0$ otherwise U(1) local symmetry breaks down.

The first and the second terms describe a spin zero (scalar) particle with 2 DoF and mass m.

But our demand to have $\mathcal{L}(x)$ with local gauge invariance leads to more predictions:

Scalar Lagrangian with local U(1) Symmetry

$$\mathcal{L}_{S=0} = \frac{1}{2}(\partial_\mu + ieA_\mu)\Phi^*(\partial^\mu - ieA^\mu)\Phi - \frac{m^2}{2}\phi^*\phi$$

$$\mathcal{L}_{S=0} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2}{2}\phi^*\phi + \frac{1}{2}ieA_\mu\Phi^*\partial^\mu\phi + $$

$$-\frac{i}{2}eA^\mu\partial_\mu\phi^*\cdot\phi + \frac{1}{2}(-ie)(+ie)A_\mu A^\mu\phi^*\phi$$

$$\mathcal{L}_{S=0} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2}{2}\phi^*\phi + \underbrace{\frac{ie}{2}(\Phi^*\partial^\mu\phi - \Phi\partial^\mu\phi^*)A_\mu}_{J_\mu A^\mu}$$

$$+ \frac{e^2}{2}A_\mu A^\mu\Phi^*\Phi$$

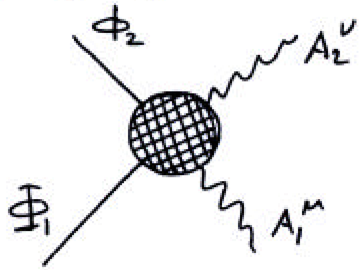
As you should by now recognise from our field theory introduction $H_I = J_\mu\cdot A^\mu + \frac{e^2}{2}A_\mu A^\mu\Phi^*\Phi$ and the transition amplitude will have a term $e^{i\int d^4x\,\overbrace{(J_\mu A^\mu + e^2/2\, A_\mu A^\mu\phi^*\phi)}^{H_I}}$

So by expanding this one gets

$$1 + i\int d^4x\left[J_\mu A^\mu + \frac{e^2}{2}A_\mu A^\mu\phi^*\phi\right] + .$$

It appears that our Lagrangian describes the Electrodynamics of massive Scalar fields. Or put in a different way: It describes the interactions of a massless vector field, the photon, with massive scalar and charged fields. This can be shown as follows:

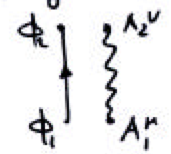
Consider the interaction of a photon with the charged scalar particle:



As we have already learned in Field theory the Scattering amplitude will be:

$$A = \langle 0|T\left\{\Phi_2\Phi_1 A_1^\mu A_2^\nu\left(1 + i\int d^4x\left[J_\rho A^\rho + \frac{e^2}{2}A_\rho A^\rho\phi^*\phi + ..\right]\right)\right\}|0\rangle$$

The e^0 term gives disconnected diagrams:



which have zero amplitude

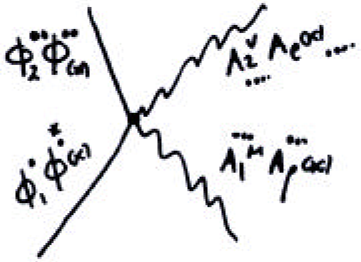
The Scalar QED Feynman Diagrams

(61)

The e^2 term will give:

$$\langle 0 | T\left(\phi_1 \phi_2 A_1^\mu A_2^\nu \left(\tfrac{i}{2} e^2 A_\rho(x) A^\rho(x) \Phi^*(x) \Phi^*(x)\right)\right) | 0 \rangle$$

$$\Phi_1^{\circ} \, \Phi^{*\circ}(x) \quad \phi_2^{\circ\circ} \, \Phi^{*\circ\circ}(x) \quad A_1^{\overset{\circ\circ\circ}{\mu}} \, A_\rho(x) \quad A_2^{\overset{\circ\circ\circ\circ}{\nu}} \, A^\rho(x)$$

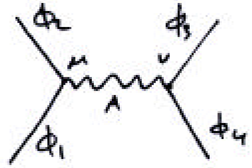


So our theory predicts a "point like" Compton scattering with strength e^2 and if you carry the full calculation you will find that the Feynman rule for this diagram is

$$2ie^2 g_{\mu\nu}$$

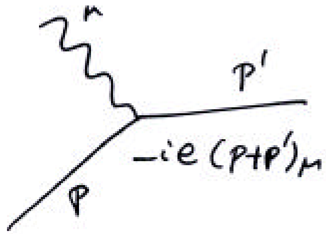
(62)

In a similar way the $J_\mu A^\mu$ term which is order-e will result to e^2 terms corresponding to diagrams that look like:



order e^2

and correspond to the Feynman rule



So the amplitude to first order perturbation will go like e^2 just like in QED but the Feynman rules are different

Conclusions

Conclusion (63)

1. Consider: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi^* \partial^\mu\phi - \frac{m^2}{2}\phi^*\phi$

 It describes a scalar ($s=0$) charged field.

2. Demand that $\mathcal{L}$ is invariant under

$$\Phi \to \Phi' = e^{ie\alpha(x)}\Phi(x) \quad U(1) \text{ group}$$

3. To do this you have to introduce:

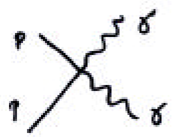
$$A_\mu, \; D_\mu = \partial_\mu - ieA_\mu$$

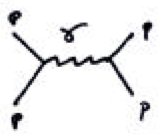
and $$A'_\mu = A_\mu + \partial_\mu \alpha(x)$$

4.- The Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{m^2}{2}\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

5.- It predicts: (a) $m_\gamma = 0$

(b) 

(c) 

And all these by just requiring that the Lagrangian is invariant under LOCAL GAUGE TRANS.

Non-Abelian Symmetries

Non-Abelian Transformations (64)

Last time we worked out an example with the $U(1)$ group which is an abelian group. That is, the group operations commute: $e^{ia(x)} \cdot e^{ib(x)} = e^{ib(x)} \cdot e^{ia(x)}$

Today we will consider groups where the group operations do not commute.

Consider for example the $SU(2)$ group. Under this group fields transform as

$$\Phi \to \Phi' = e^{-i \frac{\vec{\tau}}{2} \cdot \vec{\theta}} \Phi \quad (A)$$

where $\vec{\tau} = \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices and $\vec{\theta} = \vec{\theta}(x)$ is an x-dependent vector.

under (A) $\delta \Phi = -\frac{i}{2} (\vec{\tau} \cdot \vec{\theta}) \cdot \Phi$

$$= -\frac{i}{2} (\vec{\tau} \cdot \vec{\theta})_{ij} \Phi_j$$

(65)

Were Φ_j $j=1,2$ are two complex fields forming an $SU(2)$ doublet:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

The Lagrangian:

$$\mathcal{L} = \frac{1}{2} [\partial_\mu \phi]^\dagger \partial^\mu \phi - \frac{m^2}{2} \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2$$

has the $SU(2)$ global symmetry.

As before we would like to "gauge" this symmetry and make the Lagrangian invariant also under local $SU(2)$ $\{\vec{\theta} = \vec{\theta}(x)\}$.

As we did with $U(1)$ symmetry we require that $\partial_\mu \to D_\mu = \partial_\mu - i g \vec{\tau} \cdot \vec{A}_\mu$.

Note # of fields A^i_μ = # of group generators

In general: $SU(N) \Rightarrow N^2 - 1$ generators

$O(N) \Rightarrow \frac{N}{2}(N-1)$ generators

$\Rightarrow SU_c(3) \to 8$ generators, $SU(2) \to 3$ generators

Gauging a Non-Abelian Theory

(66)

This is the reason that the mediators of the weak interactions are three ($W^\pm, Z^0$) and also for having 8 different gluons in QCD.

——— ★ ———

To "gauge" the theory and make the Lagrangian invariant under SU(2) local transformations require as with U(1):

$$D'_\mu \Phi'(x) = D'_\mu U(\vec{\theta})\Phi(x) = U(\vec{\theta}) D_\mu \Phi(x) \Rightarrow$$

if $D_\mu = \partial_\mu - ig\vec{\tau}\cdot\vec{A}_\mu$

$$(\partial'_\mu - ig\vec{\tau}\vec{A}'_\mu)U(\vec{\theta})\Phi(x) = U(\vec{\theta})(\partial_\mu - ig\vec{\tau}\cdot\vec{A}_\mu)\Phi$$

$$\partial_\mu U(\theta)\cdot\Phi + U(\theta)\partial_\mu\Phi - ig\vec{\tau}\vec{A}'_\mu U(\vec{\theta})\Phi(x) =$$

$$U(\theta)\partial_\mu\Phi - igU(\vec{\theta})\vec{\tau}\cdot\vec{A}_\mu\Phi \Rightarrow$$

$$\partial_\mu U(\vec{\theta}) - ig\vec{\tau}\vec{A}'_\mu U(\vec{\theta}) = -igU(\vec{\theta})\vec{\tau}\cdot\vec{A}_\mu \Rightarrow$$

$$\partial_\mu U(\vec{\theta})\cdot U^{-1}(\theta) - ig\vec{\tau}\cdot\vec{A}'_\mu = -igU(\vec{\theta})\vec{\tau}\cdot\vec{A}_\mu U^{-1}(\vec{\theta})$$

(67)

$$\rightarrow ig\vec{\tau}\vec{A}'_\mu = igU(\vec{\theta})\vec{\tau}\cdot\vec{A}_\mu U^{-1}(\vec{\theta}) + \partial_\mu U(\vec{\theta})U^{-1}(\vec{\theta})$$

$$\boxed{\vec{\tau}\cdot\vec{A}'_\mu = U(\vec{\theta})\vec{\tau}\cdot\vec{A}_\mu U^{-1}(\theta) - \frac{i}{g}\partial_\mu U(\vec{\theta})U^{-1}(\vec{\theta})}$$

Define
$$\mathbb{A}'_\mu = \vec{\tau}\cdot\vec{A}'_\mu = \tau^1 A^{1'}_\mu + \tau^2 A^{2'}_\mu + \tau^3 A^{3'}_\mu$$
$$\mathbb{A}_\mu = \vec{\tau}\cdot\vec{A}_\mu = \tau_1 A^1_\mu + \tau^2 A^2_\mu + \tau^3 A^3_\mu$$

Then we have that:

$$\boxed{\mathbb{A}'_\mu = U(\theta)\mathbb{A}_\mu U^{-1}(\theta) - \frac{i}{g}\partial_\mu U(\vec{\theta})\cdot U^{-1}(\theta)} \quad (G)$$

Summary:

If $\Phi \rightarrow \Phi' = U(\vec{\theta})\Phi = e^{-i\frac{\vec{\tau}}{2}\cdot\vec{\theta}}\Phi$ and

$$\mathbb{A}'_\mu = U(\theta)\mathbb{A}_\mu U^{-1}(\vec{\theta}) - \frac{i}{g}\partial_\mu U(\theta)\cdot U^{-1}(\theta)$$

Then the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi)$$

is invariant under local gauge transf. based on the SU(2) group.

The Lagranian for a Non-Abelian Gauge Field

(68)

But the kinetic term for the gauge fields is still missing:

Define $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\,\epsilon^{ijk} A^j_\mu A^k_\nu$

and $\mathbb{F}_{\mu\nu} = T^i F^i_{\mu\nu}$

Then as you will show for Homework II the Lagrangian $\mathcal{L} = -\frac{1}{4} Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})$ is invariant under (G).

Therefore the full Lagrangian of the theory is:

$$\mathcal{L} = \mathcal{L}_{GAUGE} + \mathcal{L}_{Scalar} \rightarrow:$$

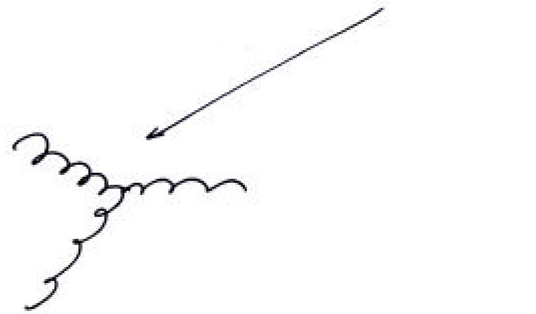
$$\mathcal{L} = -\frac{1}{4} Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}) + \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^\dagger\phi)$$

(69)

Note: (a) $[T^i T^j] = i\,\epsilon^{ijk} T^k$

$Tr(T^i T^j) = k\cdot\delta^{ij}$ k=constant

(b) $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \underbrace{g\,\epsilon^{ijk} A^j_\mu A^k_\nu}$



The $g\,\epsilon^{ijk} A^j_\mu A^k_\nu$ term which appears in non-abelian theories is responsible for these couplings which do not exist in abelian theories.

Non-Abelian Gauge Transformations

(70)

$$A'_\mu = U A_\mu U^{-1} - \frac{i}{g}\partial_\mu U \cdot U^{-1}$$

$$U(\vec\theta) = e^{-i\vec\tau\cdot\vec\theta/2} = 1 - \frac{i}{2}\vec\theta\cdot\vec\tau + \cdots$$

$$T^i A^{i\prime}_\mu = \left(1 - i/2\,\vec\theta\cdot\vec\tau\right) T^e A^e_\mu \left(1 + \frac{i}{2}\vec\theta\cdot\vec\tau\right) +$$
$$- i/g\left(-\frac{i}{2}\partial_\mu\vec\theta\cdot\vec\tau\right) U U^{-1} \rightarrow$$

$$T^i A^{i\prime}_\mu = \left(T^e A^e_\mu - i/2\,\theta^k T^k T^e A^e_\mu\right)\left(1 + i\frac{\theta^j T^j}{2}\right)$$
$$- i/g\left(-i/2\,\partial_\mu\vec\theta\cdot\vec\tau\right) \rightarrow$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu - \frac{i}{2} T^k T^e \theta^k A^e_\mu + \frac{i}{2} T^e T^j \theta^j A^e_\mu$$
$$- i/g\left(-i/2\,\partial_\mu\theta^i T^i\right)$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu + \frac{i}{2} T^e T^k \theta^k A^e_\mu - \frac{i}{2} T^k T^e \theta^k A^e_\mu$$
$$- \frac{1}{2g}\partial_\mu\theta^i T^i$$

(71)

$$\rightarrow T^i A^{i\prime}_\mu = T^i A^i_\mu + \frac{i}{2}[T^e, T^k]\theta^k A^e_\mu +$$
$$- \frac{1}{2g}\partial_\mu\theta^i T^i \rightarrow$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu + \frac{i}{2} i\,\epsilon^{ekm} T^m \theta^k A^e_\mu +$$
$$- 1/2g\,\partial_\mu\theta^i T^i$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu - \frac{1}{2}\epsilon^{mek} T^m \theta^k A^e_\mu$$
$$- 1/2g\,\partial_\mu\theta^i T^i$$

$$A^{i\prime}_\mu = A^i_\mu - \frac{1}{2}\epsilon^{iek}\theta^k A^e_\mu - \frac{1}{2g}\partial_\mu\theta^i$$

$$A^{i\prime}_\mu = A^i_\mu + \frac{1}{2}\epsilon^{ike}\theta^k A^e_\mu - \frac{1}{2g}\partial_\mu\theta^i$$

this would be the non-abelian version of $A'_\mu = A_\mu + \partial_\mu\theta$ in QED

Exercises

Exercise: Calculate $[\mathbb{D}_\mu, \mathbb{D}_\nu]\Phi$ (#2)

Hint: you should be getting the result in terms of $\mathbb{F}_{\mu\nu}$

Exercise: Show that:

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu - ig[\mathbb{A}_\mu, \mathbb{A}_\nu]$$

Spontaneously Broken Symmetries

Spontaneously broken symmetries (73)

It is possible that a given Lagrangian has a symmetry but the vacuum of the theory does not have the same symmetry. In other words the symmetry transformations which leave the Lagrangian invariant, do not leave the vacuum of this theory invariant.

As an example consider again the scalar field Lagrangian with the ϕ^4 term:

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 = T - V(\phi)$$

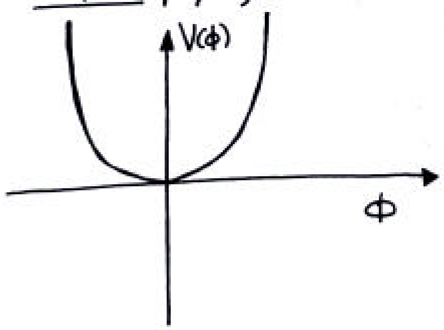
$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

It is obvious that this Lagrangian has the discrete symmetry $\Phi \to -\Phi$. That is the Lagrangian is invariant under reflection.

If $m^2>0$ and $\lambda>0$ $V(\phi)$ has a vacuum at $\langle\phi\rangle_0 = 0$. We have chosen $\lambda>0$ so that the field is bounded and we have also chosen $\mu^2>0$ in order to get the Klein-Gordon equation and interpret μ as the mass.

Suppose that $\mu^2<0$ while $\lambda>0$. Then (74) the $\mu^2/2\,\phi^2$ term is no longer a mass term and the potential $V(\phi)$ has changed in a significant way:

Before: $\mu^2>0$; $\lambda>0$



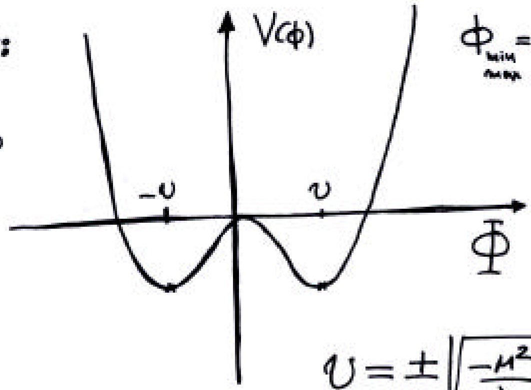
$$V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

$$\frac{dV}{d\phi} = 0 = \mu^2\phi + \lambda\phi^3 = 0$$

$$(\mu^2 + \lambda\phi^2)\phi = 0$$

$$\Phi_{\substack{min\\max}} = \begin{cases} 0 \\ \pm\sqrt{\frac{-\mu^2}{\lambda}} \end{cases}$$

After: $\lambda>0$, $\mu^2<0$



$$\upsilon = \pm\sqrt{\frac{-\mu^2}{\lambda}}$$

The origin of mass

(75)

Clearly under the reflection transformation the vacuum changes from v to $-v$ and we find ourselves in a situation where $\mathcal{L}(x)$ is invariant but $\langle\phi\rangle_0$ the vacuum expectation value is not. This way the symmetry is

SPONTANEOUSLY BROKEN!!

Perturbation theory will not work around $\phi=0$ so we redefine the field around the true vacuum 'v' and $\Phi' = \Phi - v \Rightarrow \Phi = \Phi' + v$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{\mu^2}{2}(\Phi'+v)^2 - \frac{\lambda}{4}(\Phi'+v)^4 \rightarrow$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{\mu^2}{2}(\Phi'^2+v^2+2\Phi'v) - \frac{\lambda}{4}(\Phi'^2+v^2+2\Phi'v)\cdot(\Phi'^2+v^2+2\Phi'v) \rightarrow$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi'\partial^\mu\Phi' + \Phi'\underbrace{\left(-\mu^2 v - \frac{\lambda}{4}2v^3 - \frac{\lambda}{4}2v^3\right)}_{-\mu^2 v - \lambda v^3 + \lambda v^3 - \lambda v^3 = 0} + \Phi'^2\underbrace{\left(-\frac{\mu^2}{2} - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}4v^2\right)}_{-\frac{\mu^2}{2} - \frac{\lambda}{2}v^2 - \lambda v^2} +$$

$$+ \Phi'^3\underbrace{\left(-\frac{\lambda}{4}2v - \frac{\lambda}{4}2v\right)}_{-\lambda v} - \frac{\lambda}{4}\Phi'^4$$

$\boxed{-\mu^2 = \lambda v^2}$

$\frac{\lambda v^2}{2} - \frac{\lambda v^2}{2} - \lambda v^2 = \mu^2$ (with $\frac{\lambda v^2}{2}$ terms crossed out)

(76)

And the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + \mu^2\Phi'^2 - \lambda v\Phi'^3 - \frac{\lambda}{4}\Phi'^4$$

Note that now we have a real mass term because $\mu^2 < 0$ and the $\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi'$, $\mu^2\Phi'^2$ have opposite sign $\Rightarrow$ Klein-Gordon equation.

So the term $-\frac{\mu^2}{2}\Phi^2$ goes away and instead we get a $\mu^2\Phi'^2$ term which gives us

$$\frac{(\text{mass})^2}{2} = -\mu^2 > 0 \Rightarrow \text{mass} = \sqrt{-2\mu^2}$$

So the symmetry brake leads to massive boson field which does not exhibit the symmetry of the Lagrangian.

Another Model with Spontaneously Broken Symmetry

Consider now another example which has two fields: (77)

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \partial^\mu \pi\right] - V(\sigma^2 + \pi^2)$$

$$V(\sigma^2+\pi^2) = +\frac{1}{2}\mu^2(\sigma^2+\pi^2) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2$$

The Lagrangian is invariant under $O(2)$, the group of 2-D rotations

$$\begin{pmatrix}\sigma' \\ \pi'\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\sigma \\ \pi\end{pmatrix} \quad (A)$$

or if $\theta \ll$

$$\begin{pmatrix}\sigma' \\ \pi'\end{pmatrix} \cong \begin{pmatrix}1 & \theta \\ -\theta & 1\end{pmatrix}\begin{pmatrix}\sigma \\ \pi\end{pmatrix}$$

$$\cong \left\{\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + \theta\underbrace{\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}}_{\text{group generator}}\right\}\begin{pmatrix}\sigma \\ \pi\end{pmatrix}$$

$$\left.\begin{aligned}\sigma' &= \sigma + \theta\pi \\ \pi' &= \pi - \theta\sigma\end{aligned}\right\} \Rightarrow \begin{aligned}\pi &\cong \frac{1}{1+\theta^2}(\pi' + \theta\sigma') \\ \sigma &\cong \frac{1}{1+\theta^2}(\sigma' - \theta\pi')\end{aligned}$$

The potential depends on $\sigma^2+\pi^2$ only and using (A) it is easy to show that it is $O(2)$ invariant

(78)

$$(A) \rightarrow \begin{pmatrix}\sigma \\ \pi\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\sigma' \\ \pi'\end{pmatrix}$$

$$\sigma^2+\pi^2 \rightarrow \cos^2\theta\, \sigma'^2 + \sin^2\theta\, \pi'^2 - 2\cos\theta\sin\theta\, \sigma'\pi' + \sin^2\theta\, \sigma'^2 + \cos^2\theta\, \pi'^2 + 2\sin\theta\cos\theta\, \sigma'\pi'$$

$$\rightarrow \sigma'^2 + \pi'^2$$

$\therefore\ V(\sigma^2+\pi^2)$ = invariant under $O(2)$

$$\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi = \frac{1}{2}\partial_\mu(\cos\theta\sigma' - \sin\theta\pi')\partial^\mu(\cos\theta\sigma' - \sin\theta\pi')$$

$$+\frac{1}{2}\partial_\mu(\sin\theta\sigma' + \cos\theta\pi')\partial^\mu(\sin\theta\sigma' + \cos\theta\pi')$$

$$= \frac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma' + \frac{1}{2}\partial_\mu\pi'\partial^\mu\pi'$$

Therefore the Lagrangian is indeed invariant under $O(2)$ [$=U(1)$]

Continued........

(79)

Just as we did before let $\mu^2 < 0 \quad \lambda > 0$

$$V = \frac{1}{2}\mu^2(\sigma^2+\pi^2) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2$$

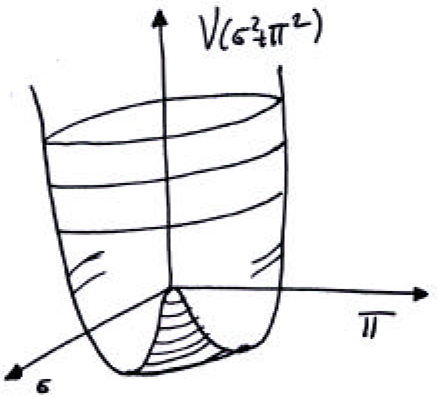
$$\frac{\partial V}{\partial \sigma} = \mu^2\sigma + \frac{\lambda}{4}2(\sigma^2+\pi^2)2\sigma = 0$$

$$= \sigma\left[\mu^2 + \lambda(\sigma^2+\pi^2)\right] = 0$$

$$\frac{\partial V}{\partial \pi} = \pi\left[\mu^2 + \lambda(\sigma^2+\pi^2)\right] = 0$$

So the potential has minimum for

$$\boxed{\sigma^2+\pi^2 = -\mu^2/\lambda}$$



(80)

Define again the radius as $v = \sqrt{-\mu^2/\lambda}$

$\therefore \quad v^2 = \sigma^2+\pi^2 = -\mu^2/\lambda$

Transform as before $\quad s = \sigma - v = \sigma - \langle\sigma\rangle_0$

$\qquad \pi = \pi$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{\mu^2}{2}(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu s\partial^\mu s + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{\mu^2}{2}(s^2+v^2+2sv+\pi^2) - \frac{\lambda}{4}(s^2+v^2+2sv+\pi^2)^2$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu s\partial^\mu s + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{\mu^2}{2}(s^2+v^2+2sv+\pi^2) - \frac{\lambda}{4}(s^2+v^2+2sv+\pi^2)(s^2+v^2+2sv+\pi^2)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu s\partial^\mu s + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + s^2\left[-\frac{\mu^2}{2} - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}4v^2\right] +$$

$$\left(\cancel{\tfrac{\lambda v^2}{2}} - \cancel{\tfrac{\lambda v^2}{2}} - \lambda v^2 = +\mu^2\right)$$

$$+\pi^2\left[-\frac{\mu^2}{2} - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}v^2\right]$$

$$\left(\lambda v^2/2 - \lambda v^2/2 = 0\right)$$

$$+ s\left[-\frac{\mu^2}{2}2v - \frac{\lambda}{4}2v^3 - \frac{\lambda}{4}2v^3\right]$$

$$\left(\lambda v^3 - \lambda v^3 = 0\right)$$

$$-\frac{\lambda}{4}\left[2s^3v + s^2\pi^2 + 2s^3v + 2sv\pi^2 + \pi^2 s^2 + 2sv\pi^2 + s^4 + \pi^4\right]$$

The Goldstone Theorem

(81)

$$\mathcal{L}=\frac{1}{2}\partial_\mu s\partial^\mu s+\frac{1}{2}\partial_\mu\pi\partial^\mu\pi+\mu^2 s^2+0\cdot\pi^2+0\cdot s$$
$$-\frac{\lambda}{4}\left\{(s^2+\pi^2)^2+4s^3v+4sv\pi^2\right\}\to$$

$$\mathcal{L}=\frac{1}{2}(\partial_\mu s\partial^\mu s+\partial_\mu\pi\partial^\mu\pi)+\mu^2 s^2+0\pi^2+0\cdot s+$$
$$-\frac{\lambda}{4}(s^2+\pi^2)^2-\lambda v s(s^2+\pi^2)$$

Interesting stuff !!!

- By choosing $s=\sigma-v$, $\pi=\pi$ we made the choice that the vacuum of the theory is $\begin{pmatrix}v\\0\end{pmatrix}$
  (we have the freedom to choose a solution of $\sigma^2+\pi^2=-\mu^2/\lambda$)
  This vacuum brakes $O(2)$ (try it if you like)
- We started with a theory with two fields σ,π and no mass term and because the theory has a broken symmetry we got one field, the S, with mass and another field, the π, massless!!!

(82)

$$M_s^2=-2\mu^2>0 \quad (\text{because } \mu^2<0)$$
$$M_\pi^2=0$$

This is not an accident and comes from the GOLDSTONE THEOREM:

"If a theory has a continuous symmetry of the Lagrangian which is not a symmetry of the vacuum then we have a massless boson in the theory"

Consider a theory that has the $O(n)$ group. The Lagrangian of such a theory will look like

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i-\frac{\mu^2}{2}\phi^i\phi^i-\frac{\lambda}{4}(\phi^i\phi^i)^2$$

or if $\Phi=\begin{pmatrix}\phi_1\\ \vdots\\ \phi_n\end{pmatrix}$ $\quad \mathcal{L}=\frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\Phi-\frac{\mu^2}{2}\tilde{\Phi}\Phi-\frac{\lambda}{4}(\tilde{\Phi}\Phi)^2$

$\tilde{\Phi}=(\phi_1 \ldots \phi_n)$

The Goldstone Theorem II

(83)

Suppose again that $\mu^2 < 0$ and therefore there is a minimum of $\frac{\partial V}{\partial \phi_i} = 0$

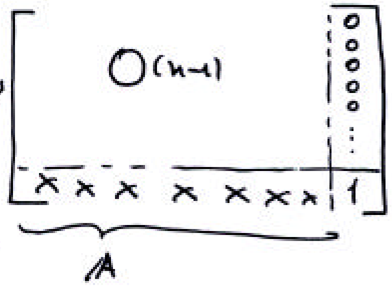
$$\Phi_e\left[\mu^2 + \lambda(\phi^i\phi^i)\right] = 0$$

So as before one of the fields will develop a vacuum expectation value (VEV).

$$\langle\phi\rangle_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}$$

$O(n)$ has $\frac{1}{2}n(n-1)$ generators. Clearly not all are broken

And transformations like this:



leave the vacuum invariant

$$A\langle\phi\rangle_0 = \langle\phi\rangle$$

So the vacuum is invariant under $O(n-1)$ but not under $O(n)$

(84)

The group $O(n-1)$ has $\frac{1}{2}(n-1)(n-2)$ generators, therefore the number of broken generators is

$$N(O(n)) - N(O(n-1)) =$$

$$\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$$

This you can see also from the previous picture: There are $n-1$ zeros which if they are replaced by something finite brake the symmetry (give another VEV)

So the most general form of our field is

$$\Phi = e^{i\xi_i \cdot K_i / v} \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ v + \eta \end{pmatrix} \qquad 1 \le i < n-1$$

$n-1$ broken generators

Conclusion

Since complex numbers are involved, I will use "†" instead of "T" in the Lagrangian. So: (85)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^\dagger(\partial^\mu \phi) - \frac{\mu^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2$$

By substituting $\Phi = e^{i\xi_i \cdot k_i/v}\begin{pmatrix} 0 \\ \vdots \\ n+v \end{pmatrix}$ into the Lagrangian (Keep in mind the form of k_i) you get

$$\mathcal{L} = \frac{1}{2}(\partial_\mu n\, \partial^\mu n + \partial_\mu \xi^i \partial^\mu \xi^i) - \frac{\mu^2}{2}(n+v)^2 + -\frac{\lambda}{4}(n+v)^4 + H.O.T$$

- The n field again develops a VEV and the $n-1$ ξ_i remain massless!!!

Bottom line: THE NUMBER OF THE BROKEN GENERATORS IS EQUAL TO THE NUMBER OF MASSLESS GOLDSTONE BOSONS THAT EXIST IN THIS THEORY

Example: The Complex Scalar Lagrangian

(86)

We can demonstrate all these for n=2. Go back to the Lagrangian of the σ,π model and write the same Lagrangian in complex form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \mu^2/_2\,\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \;;\mu^2<0$$

Write $\Phi = \rho\, e^{i\theta/v}$ ρ(x), θ(x) are fields

$$v = \sqrt{\frac{-\mu^2}{\lambda}}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu(\rho e^{-i\theta/v})\partial^\mu(\rho e^{i\theta/v}) - \mu^2/_2\,\rho^2 - \frac{\lambda}{4}\rho^4$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\rho e^{-i\theta/v} - i\frac{\partial_\mu\theta}{v}\rho e^{-i\theta/v}\right)\left(\partial^\mu\rho e^{i\theta/v} + i\frac{\partial^\mu\theta}{v}\rho e^{i\theta/v}\right) +$$

$$-\frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\frac{1}{v^2}\partial_\mu\theta\partial^\mu\theta\,(\rho)^2 - \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

go to the minimum and expand $\rho = n + v$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho\partial^\mu\rho) + \underbrace{\frac{1}{2v^2}\partial_\mu\theta\partial^\mu\theta\,(n+v)^2}_{\underbrace{\frac{1}{2v^2}\partial_\mu\theta\partial^\mu\theta\,(n^2+v^2+2nv)}_{\frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \text{H.O.T}}} - \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

(87)

And the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\partial_\mu\theta\partial^\mu\theta - \frac{\mu^2}{2}(n^2+v^2+2nv) +$$

$$-\frac{\lambda}{4}(n^2+v^2+2nv)$$

Observations: ① No $\frac{m^2}{2}\theta^2$ term ⇒ $\boxed{m_\theta = 0}$ and θ is a Goldstone Boson

② The coefficient of n^2 is $\left[-\frac{\mu^2}{2} - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}4v^2 - \frac{\lambda}{4}v^2\right]$

$$= \mu^2 < 0$$

$$\Rightarrow m^2_{mass}(n) = -2\mu^2$$

$$\boxed{m_n^2 = -2\mu^2 > 0}$$

∴ The theory has one Goldstone boson ($m_\theta = 0$) because one generator is broken and one massive scalar (boson).

Higgs Mechanism

Higgs mechanism

(88)

Consider the same theory as before but now require that the theory is gauged. That is, it is invariant under local U(1) transform.

$$D_\mu = \partial_\mu - ieA_\mu \qquad A_\mu' = A_\mu - \partial_\mu \alpha(x)$$
$$U = e^{-ie\alpha(x)} \qquad \Phi' = U\Phi$$

$$\mathcal{L} = \frac{1}{2}(D_\mu\Phi)^* D^\mu \Phi - \frac{m^2}{2}\Phi^*\Phi - \frac{\lambda}{4}(\Phi^*\Phi)^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu + ieA_\mu)\rho e^{-i\theta}(\partial^\mu - ieA^\mu)\rho e^{i\theta} + - \frac{m^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

$$\Phi = \rho e^{i\theta}$$

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu\rho\, e^{-i\theta} + \rho(-i\partial_\mu\theta)e^{-i\theta} + ieA_\mu\rho e^{-i\theta}\right\}\cdot$$
$$\left\{\partial_\mu\rho e^{i\theta} + i\partial_\mu\theta\,\rho e^{i\theta} - ieA^\mu\rho e^{i\theta}\right\} +$$
$$- \frac{m^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

(89)

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu\rho + \rho\cdot i\cdot(A_\mu\cdot e - \partial_\mu\theta)\right\}\cdot$$
$$\left\{\partial^\mu\rho - i\rho(A^\mu\cdot e - \partial_\mu\theta)\right\} - \frac{m^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

Call: $\Theta = \frac{1}{e}\theta$

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu\rho + ie\rho(A_\mu - \partial_\mu\Theta)\right\}\cdot$$
$$\left\{\partial^\mu\rho - ie\rho(A_\mu - \partial_\mu\Theta)\right\} - \frac{m^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

Call: $A_\mu' = A_\mu - \partial_\mu\Theta$

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu\rho + ie\rho[A_\mu']\right\}\left\{\partial^\mu\rho - ie\rho[A_\mu']\right\} +$$
$$- \frac{m^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

So what just happened is that the vector boson A_μ which had no mass and two degrees of freedom (2-transv. polarizations)

The Goldstone Bosons Disappear…

ate the Goldstone Boson θ that (90)
had no mass and one degree of freedom and now we have a massive vector Boson with 3 degrees of freedom (two transverse and one Longitudinal polarizations)

So $\mathcal{L} = \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + e^2\rho^2 A'_\mu A^{\mu\prime} - \frac{m^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4 +$

$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (added so that A_μ has a kinetic term also)

$\rho = n + v$ (same trick)

$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\rho\partial^\mu\rho}_{1/2\,\partial_\mu n\partial^\mu n} + e^2(n+v)^2 A'_\mu A'^\mu - \frac{m^2}{2}(n^2+v^2+2nv) +$

$- \frac{\lambda}{4}(n^2+v^2+2nv)(n^2+v^2+2nv)$

$e^2(n^2+v^2+2nv)A'_\mu A^{\mu\prime} = v^2 e^2 A'_\mu A^{\mu\prime} + HOT$

$\therefore\ m_A^2/2 = e^2v^2$

$$m_A^2 = 2e^2v^2$$

(91)
As for n, it gets its mass just as before from the n^2 coefficient of the $-\frac{m^2}{2}(n+v)^2 +$
$-\frac{\lambda}{4}(n+v)^4$ terms and

$$m_n^2 = -2m^2 > 0$$

So: The Goldstone Boson gets "eaten" by the vector boson and the vector boson gets mass and 3 degrees of freedom.
The second scalar becomes a massive one.

BEFORE HIGGS		AFTER HIGGS	
ρ	1 DoF	ρ	1 DoF
θ	1 DoF	A_μ	3 DoF
A_μ	2 DoF		4 DoF
	4 DoF		

The Weinberg Salam Model

If you do not know why the coefficient of (92) $A_\mu A^\mu$ is equal to the $\frac{m_A^2}{2}$, here is the reason: Consider the Lagrangian for $m\neq 0$ S=1 field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} = -F^{\mu\nu} \qquad \frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu$$

$$\therefore\ -\partial_\mu F^{\mu\nu} - m^2 A^\nu = 0 \Rightarrow$$

$$-\Box A^\nu + \partial^\nu(\partial_\mu A^\mu) - m^2 A^\nu = 0$$

I can always choose a gauge such that $\partial_\mu A^\mu = 0$

$\partial_\mu A'^\mu = \partial_\mu A^\mu - \Box\Theta$

$$\therefore (\Box + m^2)A^\nu = 0$$

is the equation for a massive vector field

The Weinberg Salam Model (93)

It is based on the $SU(2)_L \otimes U(1)_Y$ group:

Recall that SU(2) has 3 generators and U(1) has one. So there are 4 gauge fields $A^1_\mu, A^2_\mu, A^3_\mu, B_\mu$. And the covariant derivative can be written as:

$$D_\mu = \partial_\mu - \frac{i}{2}g'YB_\mu - \frac{i}{2}gY\vec{\tau}\vec{A}_\mu$$

For the moment take $Y=-1$ for the B_μ term and $Y=+1$ for the A_μ terms (will explain later):

$$\boxed{D_\mu = \partial_\mu + \frac{i}{2}g'B_\mu - \frac{i}{2}g\vec{\tau}\cdot\vec{A}_\mu}$$

This derivative acts on higgs doublets and Lepton doublets:

$$\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$$

Recall that $e_L = \frac{1}{2}(1-\gamma_5)e$; $e_R = \frac{1}{2}(1+\gamma_5)e$

The Weinberg Salam Lagrangian

(94)

- Therefore one can write the gauge field Lagrangian

$$\mathcal{L}_G = -\frac{1}{4} B_{\mu\nu}B^{\mu\nu} - \frac{1}{4} Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\mu A^i_\nu + g\epsilon^{ijk} A^j_\mu A^k_\nu$$

$$\mathbb{F}_{\mu\nu} = T^i F^i_{\mu\nu}$$

- The Lepton field Lagrangian can be written in terms of the doublet and singlet terms

$$\mathcal{L} = \underbrace{\bar{R} i\gamma^\mu(\partial_\mu + ig' B_\mu) R}_{\text{Singlets}} +$$

$$\bar{L} i\gamma^\mu(\partial_\mu + \frac{i}{2} g' B_\mu - ig \frac{\vec{\tau}}{2}\cdot\vec{A}_\mu) L$$

Whatever happens this theory has to have a photon with $m_\gamma = 0$ at the end. So one of the generators will leave the vacuum invariant and there will be a massive scalar coming also from the unbroken generator.

(95)

Three generators have to be broken to give us 3 Goldstone Bosons which will give mass to the remaining 3 gauge fields.

Define the complex Higgs field as

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{(4 degrees of freedom)}$$

as an SU(2) doublet and

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi^\dagger\phi)$$

$$\mathcal{L}_{Higgs} = (\partial_\mu \phi^\dagger + i g'/2\, B_\mu \mathbb{1}\phi^\dagger + i \frac{g}{2}\vec{\tau}\cdot\vec{A}_\mu \phi^\dagger)\cdot$$

$$(\partial_\mu \phi - i g'/2\, B_\mu \phi^\dagger \mathbb{1} - i \frac{g}{2}\vec{\tau}\cdot\vec{A}_\mu \phi) +$$

$$- V(\phi^\dagger\phi)$$

Aims: From 1+3 massless gauge fields (B_μ, A^i_μ) we want $(W^\pm, Z^0)$, $A_\mu \Rightarrow \mathcal{L}_{HIGGS}$ has to give us 3 Goldstone bosons + H^0 ($m \neq 0$)

WS continued…

(96)

Take $V = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$

and if you like you could also have

Couplings $\mathcal{L}_I = -G_e [\bar{R}\phi^\dagger L + \bar{L}\phi R]$

which are gauge invariant and will deal with them later.

As before you can show that if $\mu^2 < 0$

$$V(\phi^\dagger\phi) = \mu^2(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) + \lambda(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2)^2$$

if $\begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} = \begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix}$ and $\frac{\partial V}{\partial \phi_i} = 0$

and $\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v\end{pmatrix}$ $v = \sqrt{\frac{-\mu^2}{\lambda}}$

This vacuum breaks both the SU(2) and the $U(1)_Y$ symmetries (try it)

$Q = T_L^3 + \frac{1}{2}Y$ is not broken

(Charge is conserved) $\Rightarrow$ massless photon + massive scalar Higgs

(97)

Back to the covariant derivative:

$$D_\mu^\dagger = \partial_\mu + \frac{i}{2} g T^e A^e_\mu + i g'/_2 B_\mu \rightarrow$$

$$D_\mu^\dagger = \partial_\mu + i\frac{g}{2}\begin{pmatrix}A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3\end{pmatrix} + i\,g'/_2\begin{pmatrix}B_\mu & 0 \\ 0 & B_\mu\end{pmatrix} \rightarrow$$

$$D_\mu^\dagger = \partial_\mu + \frac{i}{2}\begin{pmatrix}g'B_\mu + gA_\mu^3 & (A_\mu^1 - iA_\mu^2)g \\ (A_\mu^1 + iA_\mu^2)g & g'B_\mu - gA_\mu^3\end{pmatrix}$$

$$(D_\mu\phi)^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ \partial_\mu h\end{pmatrix} + \frac{i}{2\sqrt{2}}\begin{pmatrix}g(A_\mu^1 - iA_\mu^2)(h+v) \\ (g'B_\mu - gA_\mu^3)(h+v)\end{pmatrix}$$

We have taken the field to be $\phi = e^{\frac{i\vec{\xi}\cdot\vec{\tau}}{2v}}\begin{pmatrix}0 \\ \frac{h+v}{\sqrt{2}}\end{pmatrix}$

3 broken generators

3 Goldstone bosons

At the end

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{8}(h+v)^2 g^2 (A_\mu^1 + iA_\mu^2)(A_\mu^1 - iA_\mu^2)$$
$$+ \frac{1}{8}(h+v)^2(B_\mu g' - gA_\mu^3)^2$$

Define $W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$

$$Z_\mu = \frac{-gA_\mu^3 + g'B_\mu}{\sqrt{g^2 + g'^2}}\ ;\ A_\mu = \frac{g'A_\mu^3 + gB_\mu}{\sqrt{g^2 + g'^2}}$$

The W? , Z^0 and H^0 Masses

(98)

Therefore

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}\partial_\mu h\partial^\mu h +$$
$$+\frac{1}{8}g^2(h+v)^2 2\, W_\mu^+W^{\mu-} +$$
$$+\frac{1}{8}(h+v)^2\left(\sqrt{g^2+g'^2}\right)^2 Z^0_\mu Z^{0\mu}$$

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{4}g^2(h+v)^2 W_\mu^+W^{-\mu} +$$
$$+\frac{1}{8}(h+v)^2(g^2+g'^2)\, Z^0_\mu Z^{0\mu}$$

$$\frac{1}{2}m_W^2 = +\frac{g^2}{8}v^2 \Rightarrow m_W^2 = \frac{g^2v^2}{4} \Rightarrow \boxed{M_W = \frac{gv}{2}}$$

$$\frac{1}{2}m_Z^2 = \frac{1}{8}(g^2+g'^2)v^2 \Rightarrow M_Z^2 = (g^2+g'^2)\frac{v^2}{4} \Rightarrow$$

$$\boxed{M_{Z^0} = \frac{1}{2}v\sqrt{g^2+g'^2}}$$

$$\therefore \boxed{\frac{M_W}{M_{Z^0}} = \frac{g}{\sqrt{g'^2+g^2}}}$$

(99)

Next we evaluate the Higgs mass:

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \mu^2\frac{1}{2}(h+v)^2 - \frac{\lambda}{4}(h+v)^4 \rightarrow$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{\mu^2}{2}h^2 - \frac{\lambda}{4}(h^2+2hv+v^2)(h^2+2hv+v^2)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{\mu^2}{2}h^2 - \frac{\lambda}{4}(h^2v^2+4h^2v^2+h^2v^2)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{h^2}{2}\left(\mu^2+\frac{\lambda}{2}\{v^2+4v^2+v^2\}\right)$$

$$-\frac{h^2}{2}(-\lambda v^2+3\lambda v^2)$$

$$-\lambda\frac{h^2}{2}v^2 2$$

$$-\frac{h^2}{2}(2\lambda v^2)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - (2\lambda v^2)\frac{h^2}{2} + \cdots$$

$$\boxed{M_{H^0} = 2\lambda v^2 = -2\mu^2}$$

The Photon and Z^0 Fields

(100)

Define $g'/g = \tan\theta_W$

Then

$$A_\mu = \cos\theta_W B_\mu + \sin\theta_W A_\mu^3$$

$$Z_\mu^0 = \sin\theta_W B_\mu - \cos\theta_W A_\mu^3$$

and

$$B_\mu = \cos\theta_W A_\mu + \sin\theta_W Z_\mu$$

$$A_\mu^3 = \sin\theta_W A_\mu - \cos\theta_W Z_\mu$$

WS-Model Predictions: Higgs Sector

(101)

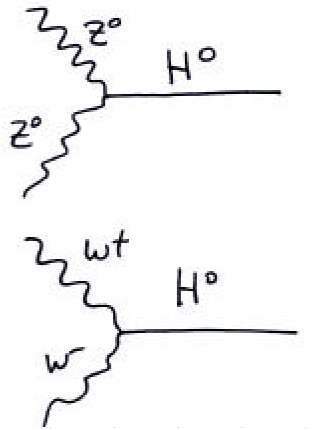
But there is more we can get from this Lagrangian:

$$\mathcal{L}_{Higgs} = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{4} g^2 (h+v)^2 W_\mu^+ W^{-\mu} + \frac{1}{8}(h+v)^2 (g^2+g'^2) Z_\mu^0 Z^{0\mu} + V(h)$$

1.– the linear terms in h: $\frac{1}{4} g^2 2hv\, W_\mu^+ W^{-\mu}$

$\frac{1}{8}(g^2+g'^2) 2hv\, Z_\mu Z^\mu$

predict



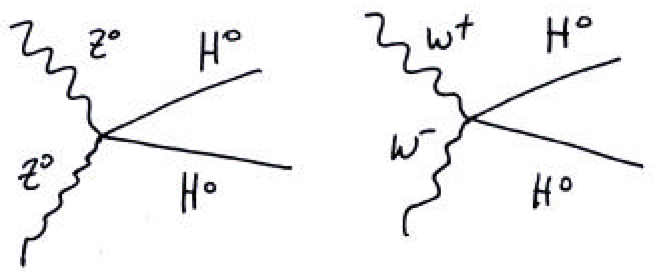
(102)

2.– The theory predicts no "pointlike" Couplings of the Higgs to photons!! (later we will see that it only goes through quark loops)

3.– The quadratic terms

$$\frac{1}{4} g^2 h^2 W_\mu^+ W^{-\mu} + \frac{1}{8}(g^2+g'^2) h^2 Z_\mu^0 Z^{0\mu}$$

predict:

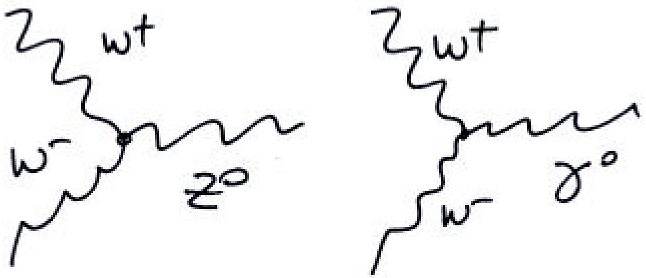


4.– $m_\gamma = 0$

Introducing Leptons in the WS Model

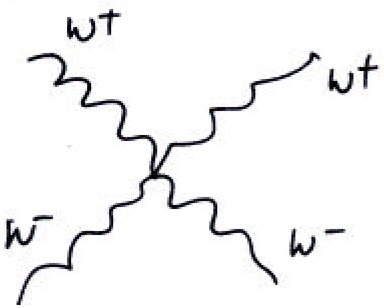
(103)

5.- Due to the non-abelian nature of our theory we get terms of the type



They come from the $g\epsilon^{ijk}A^j_\mu A^k_\nu$ term of $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\,\epsilon^{ijk}A^j_\mu A^k_\nu$ when these terms are multiplied by $\partial_\mu A^i_\nu$ in the gauge Lagrangian.

Clearly you could also have



type of terms

INTRODUCING LEPTONS IN THE THEORY (104)

$L = \begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix}$ is an $SU(2)_L$ doublet with weak hypercharge $\boxed{Y_W = -1}$

$R = e_R$ is an $SU(2)_L$ Singlet with $\boxed{Y_W = -2}$

$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ with $\boxed{Y_W = +1}$ also an $SU(2)_L$ doublet

$SU(2)_L \otimes U_Y(1)$ implies $D_\mu = \partial_\mu - i\frac{g'}{2}Y_W B_\mu - i\frac{g}{2}\vec{\tau}\cdot\vec{A}_\mu$

$$\chi_L \rightarrow \chi_L' = e^{i\beta(x)Y + i\vec{\alpha}\cdot\vec{\tau}}\,\chi_L \quad (1)$$

$$\psi_R \rightarrow \psi_R' = e^{i\beta(x)Y}\,\psi_R \quad (2)$$

The vacuum $\langle\phi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \eta + v \end{pmatrix}$ brakes all the $SU(2)$ generator and also the $U_{Y(1)}$

Charge Conservation

(105)

Try $\Phi_0 \to \Phi_0' = e^{i g/2 \vec{\alpha}\cdot\vec{\tau}} \Phi_0$ and you will find that all 3 are broken (the vacuum is not invariant). Same is true for $U_Y(1)$.

But since $1, \sigma^i$ form a basis I am going to redefine the generators in an effort to see if any combination is unbroken (need $m_\gamma = 0$ no matter what)

Try $\hat{Q} = \hat{T}_3 + \frac{1}{2}\hat{Y}$

$$\hat{Q} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{Y}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \Rightarrow$$

$$\hat{Q}\Phi_0 \overset{Y=+1}{=} \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ \frac{u+v}{\sqrt{2}}\end{pmatrix} = 0!$$

$\therefore$ $e^{iY\beta(x)}$ can be redefined to be $e^{i\hat{Q}\beta(x)}$ $\therefore \delta\Phi_0 = i\beta(x)\hat{Q}\Phi_0 = 0$

$\therefore$ $\delta\Phi_0 = 0 \Rightarrow$ one unbroken generator has been found

(106)

Consider the effect of $\hat{Q}$ on a lepton doublet:

(a) $\hat{Q}\chi_L = \left\{\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix} + \begin{pmatrix}-1/2 & 0\\ 0 & -1/2\end{pmatrix}\right\}\chi_L$

$\boxed{Y=-1}$

$$= \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu_L\\ e_L^-\end{pmatrix} = \begin{pmatrix}0\,\nu_L\\ -1\,e_L^-\end{pmatrix}$$

(b) $\hat{Q}\cdot R = \hat{Q}\,R = (0 + \frac{1}{2}(-2))\,e_R = -e_R$

$\boxed{Y=-2}$

So it is a charge operator and $\hat{Q} = T_3 + \frac{Y}{2}$ is the generator of $U_{em}(1)$ (electromagnetic)

Fortunately or unfortunately this also means that along with a massless photon the theory predicts also a massive scalar left to be found by the exp.

The Lepton couplings to W?

(104)

So far we have

Field	T_3	Y	Q	$SU(2)_L$
ν_τ, ν_e, ν_μ	$1/2$	-1	0	upper part of a doublet
e^-, μ^-, τ^-	$-1/2$	-1	-1	Lower part of a doublet
H^0	$-1/2$	$+1$	0	Lower part of a doublet
e_R, μ_R, τ_R	0	-2	-1	Singlets
Quarks				

Lets write the lepton Lagrangian:

$$\mathcal{L} = \bar{R} i \gamma^\mu \underbrace{(\partial_\mu + i g' B_\mu)}_{Y_w = -2} R + \bar{L} i \gamma^\mu \underbrace{(\partial_\mu + \tfrac{i}{2} g' B_\mu - i g \tfrac{\vec{\tau} \cdot \vec{A}}{2})}_{Y = -1} L$$

Look at the A^1_μ, A^2_μ couplings

$$\frac{g}{2} \bar{L} \gamma^\mu (T^1 A^1_\mu + T^2 A^2_\mu) L = \frac{g}{2} \times$$

$$(\bar{\nu} \; \bar{e}_L) \gamma^\mu \begin{pmatrix} 0 & A^1_\mu - i A^2_\mu \\ A^1_\mu + i A^2_\mu & 0 \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$= \frac{g}{2} (\bar{\nu} \; \bar{e}_L) \gamma^\mu \begin{pmatrix} \sqrt{2} W^+_\mu e_L \\ \sqrt{2} W^-_\mu \nu_L \end{pmatrix}$$

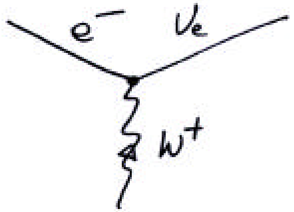
(108)

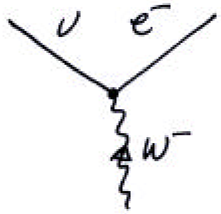
So $\mathcal{L}_I(W) =$

$$\frac{g}{2} \bar{L} \gamma^\mu (T^1 A^1_\mu + T^2 A^2_\mu) L =$$

$$\frac{g}{\sqrt{2}} \left\{ (\bar{\nu} \gamma^\mu e_L) W^+_\mu + (\bar{e}_L \gamma^\mu \nu_L) W^-_\mu \right\}$$

which predicts that the $W^\pm$ couple to leptons via the weak interaction





If you compare with the old weak interaction theory based on the G_F coupling you get

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2} = \frac{1}{2v^2}}$$

The ? and Z? couplings to Leptons

(109)

Do the same for γ, Z°

$$\mathcal{L}_I(A,Z) = -g'\bar{R}\gamma^\mu R B_\mu + \frac{g}{2}\bar{L}\gamma^\mu \tau^3 L A^3_\mu - \frac{g'}{2}\bar{L}\gamma^\mu L B_\mu \rightarrow$$

$$\mathcal{L}_I(A,Z) = -g'\bar{e}_R\gamma^\mu e_R B_\mu + \frac{g}{2}(\bar{\nu}\ \bar{e})\gamma^\mu\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\nu\\e\end{pmatrix}A^3_\mu - \frac{g'}{2}(\bar\nu\gamma^\mu\nu + \bar e\gamma^\mu e)B_\mu$$

$$\mathcal{L}_I(A,Z) = -\frac{g'}{2}\{\underbrace{2\bar e_R\gamma^\mu e_R + \bar\nu\gamma^\mu\nu_L + \bar e_L\gamma^\mu e_L}_{X}\}B_\mu + \frac{g}{2}[\underbrace{\bar\nu\gamma^\mu\nu - \bar e_L\gamma^\mu e_L}_{Y}]A^3_\mu$$

$$\mathcal{L}_I(A,Z) = -\frac{g'}{2}X\cdot(\cos\theta_W A_\mu + \sin\theta_W Z_\mu) + \frac{g}{2}Y(\sin\theta_W A_\mu - \cos\theta_W Z_\mu)$$

(110)

$$\mathcal{L}_I = \Big\{\underbrace{\cos\theta_W}_{\frac{g}{\sqrt{g^2+g'^2}}}\frac{(-g')}{2}[2\bar e_R\gamma^\mu e_R + \bar\nu\gamma^\mu\nu_L + \bar e_L\gamma^\mu e_L] + \frac{g}{2}[\bar\nu\gamma^\mu\nu - \bar e_L\gamma^\mu e_L]\underbrace{\sin\theta_W}_{\frac{g'}{\sqrt{g^2+g'^2}}}\Big\}A_\mu +$$

nice that they go away otherwise we would have νγ coupling....

$$+Z_\mu\Big\{-\frac{g'}{2}\sin\theta_W(X) - \frac{g}{2}\cos\theta_W(Y)\Big\}\rightarrow$$

$$\mathcal{L}_I(A,Z) = -\frac{gg'}{\sqrt{g^2+g'^2}}[\bar e_R\gamma^\mu e_R + \bar e_L\gamma^\mu e_L]A_\mu$$

$$+Z_\mu\Big\{-\frac{g'^2}{2\sqrt{g^2+g'^2}}[X] - \frac{g^2}{2\sqrt{g^2+g'^2}}[Y]\Big\}$$

So
$$\boxed{e = \frac{gg'}{\sqrt{g^2+g'^2}}}$$

Predictions:

$$\mathcal{L}_I = \underbrace{-e\left[\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L\right]}_{QED} + \qquad (III)$$

$$+ Z^0_\mu \left\{ \frac{-g'^2}{2\sqrt{g'^2+g^2}} \left(2\bar{e}_R \gamma^\mu e_R + \bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L\right) - \frac{g^2}{2\sqrt{g'^2+g^2}} \left(\bar{\nu} \gamma^\mu \nu - \bar{e}_L \gamma^\mu e_L\right) \right\}$$

Predictions so far:

1.- We get a photon that couples to charge and therefore we are consistent with the QED.

2.- Z^0 couples to both ν and e^-

3.- the photon does not couple to neutrinos

Fermion Masses in the WS model

(112)

Weak interaction phenomenology:

$e = g \sin\theta_W$ $\quad e = \frac{g g'}{\sqrt{g^2+g'^2}}$

$e = g' \cos\theta_W$

$\tan\theta_W = g'/g$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W} = \frac{1}{2v^2}$$

$$M_W = \frac{38}{\sin\theta_W} \text{ GeV}$$

$$M_{Z^0} = \frac{38}{\frac{1}{2}\sin 2\theta_W} \text{ GeV}$$

Muon decay

$$\mathcal{M} = -\frac{i g^2}{16\pi^2} \bar{u}(\nu_\mu)\gamma^\mu(1-\gamma_5)u(\mu)\,\bar{u}(e)\gamma^\nu(1-\gamma_5)v(\bar{\nu}_e) \times$$

$$\times \frac{g_{\mu\nu} - k_\mu k_\nu / M_W^2}{k^2 - M_W^2}$$

(113)

For a review of the weak interactions please read chapter 12 of Halzen + Martin, which is what I covered in class (some of it)

Fermion Masses

Try $\mathcal{L}_m = -m_e \bar{\psi}\psi$.

$$-m_e \bar{e} e = -m_e \bar{e}\left(\frac{1-\gamma_5}{2} + \frac{1+\gamma_5}{2}\right)e$$

$$= -m_e(\bar{e}_L e_R + \bar{e}_R e_L)$$

↑ doublet ↑ singlet

Violates gauge invariance

Lepton Masses

(114)

Clearly we need another way:

$$\mathcal{L} = -G_e \left[(\bar{\nu}_e, \bar{e})_L \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} e_R + \bar{e}_R (\phi^-, \bar{\phi}^0) \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \right]$$

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h+v \end{pmatrix}$$

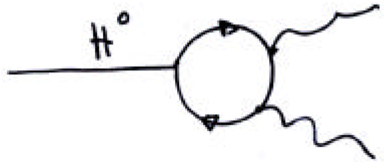
$$\mathcal{L} = -G_e \left[(\bar{\nu}_e, \bar{e})_L \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h+v \end{pmatrix} e_R + \bar{e}_R \left(0, \frac{h+v}{\sqrt{2}}\right) \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \right]$$

$$\boxed{\mathcal{L}_Y = -\frac{G_e}{\sqrt{2}} (h+v) \left[\bar{e}_L e_R + \bar{e}_R e_L \right]}$$

(115)

Therefore $m_e = -\frac{G_e v}{\sqrt{2}}$ and

$$\mathcal{L}_Y = -m_e (\bar{e}_L e_R + \bar{e}_R e_L) - \frac{m_e}{v} \bar{e} e h$$



the second term predicts that the higgs couples to leptons with a strength proportional to the lepton mass m_e.

Quarks Masses (Very Brief)

Introducing Quarks in the Standard Model (116)

Introducing quarks in the standard model is different than introducing leptons in two ways.

① If you try $\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}$ doublets you need to find a way to accomodate Cabibbo mixings

② the upper part of the multiplet needs to have mass. (By now we know that neutrinos have mass so this second difference is not valid anymore ... this way can be used to generate masses to neutrinos then)

(117)

- We do this by using in addition to the higgs doublet $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ also the SU(2) doublet $i\sigma_2\phi^* = \begin{pmatrix} \phi^0 \\ -\phi^- \end{pmatrix} = \tilde{\phi}_{\boxed{Y=-1}}$
- The quark doublets can be written as

$$N_L = \begin{pmatrix} p_L \\ n_L\cos\theta + \lambda_L\sin\theta \end{pmatrix} \equiv \begin{pmatrix} p_L \\ n_{cL} \end{pmatrix}$$

$$\lambda_C = \lambda_L\cos\theta - n_L\sin\theta$$

(assuming two quark mixing ....)

Then

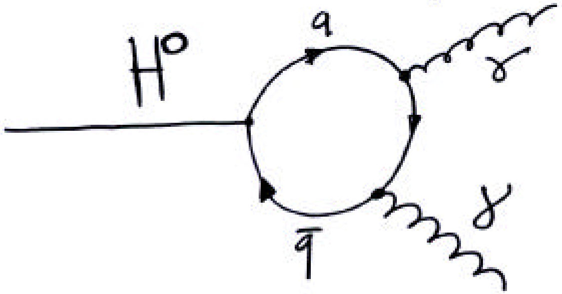
$$\mathcal{L}_Y = [\bar{N}_L\tilde{\phi}p_R + h.c.] + G_2[\bar{N}_L\phi n_R + h.c.] +$$
$$G_3[\bar{N}_L\phi\lambda_R + h.c.] + G_4[n_R\lambda_c] + G_5[\lambda_R ...$$

This again will give masses to all quarks and will predict that the Higgs couples to the quarks with couplings proportional to the quark mass:

Running out of time…..

(118)

By now it should be clear that the $H^0 \to \gamma\gamma$ channel at LHC goes through a quark loop with a coupling proportional to the quark mass



Same is true when you produce the Higgs via gg