Spontaneously Broken Symmetries

Spontaneously broken symmetries (73)

It is possible that a given Lagrangian has a symmetry but the vacuum of the theory does not have the same symmetry. In other words the symmetry transformations which leave the Lagrangian invariant, do not leave the vacuum of this theory invariant.

As an example consider again the scalar field Lagrangian with the ϕ^4 term:

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 = T - V(\phi)$$

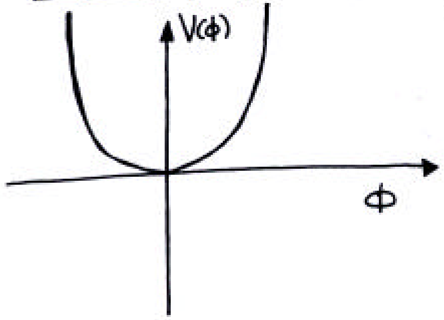
$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

It is obvious that this Lagrangian has the discrete symmetry $\Phi \to -\Phi$. That is the Lagrangian is invariant under reflection.

If $m^2>0$ and $\lambda>0$ $V(\phi)$ has a vacuum at $\langle\phi\rangle_0 = 0$. We have chosen $\lambda>0$ so that the field is bounded and we have also chosen $\mu^2>0$ in order to get the Klein-Gordon equation and interpret μ as the mass.

Suppose that $\mu^2<0$ while $\lambda>0$. Then (74) the $\mu^2/2\,\phi^2$ term is no longer a mass term and the potential $V(\phi)$ has changed in a significant way:

Before: $\mu^2>0$; $\lambda>0$

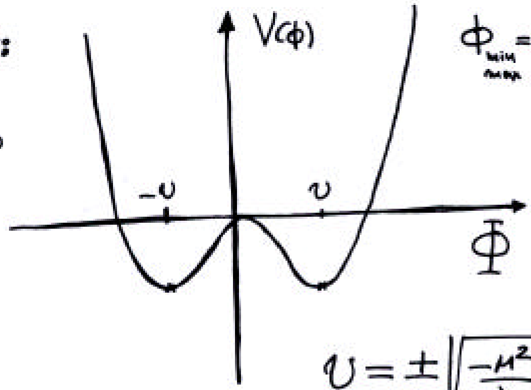


$$V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

$$\frac{dV}{d\phi} = 0 = \mu^2\phi + \lambda\phi^3 = 0$$

$$(\mu^2 + \lambda\phi^2)\phi = 0$$

After: $\lambda>0$ $\mu^2<0$



$$\Phi_{\substack{min\\max}} = \begin{cases} 0 \\ \pm\sqrt{\frac{-\mu^2}{\lambda}} \end{cases}$$

$$\upsilon = \pm\sqrt{\frac{-\mu^2}{\lambda}}$$

The origin of mass

(75)

Clearly under the reflection transformation the vacuum changes from v to $-v$ and we find ourselves in a situation where $\mathcal{L}(x)$ is invariant but $\langle\phi\rangle_0$ the vacuum expectation value is not. This way the symmetry is SPONTANEOUSLY BROKEN!!

Perturbation theory will not work around $\phi=0$ so we redefine the field around the true vacuum 'v' and $\Phi' = \Phi - v \Rightarrow \Phi = \Phi' + v$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{\mu^2}{2}(\Phi'+v)^2 - \frac{\lambda}{4}(\Phi'+v)^4 \rightarrow$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{\mu^2}{2}(\Phi'^2+v^2+2\Phi' v) - \frac{\lambda}{4}(\Phi'^2+v^2+2\Phi' v)\cdot(\Phi'^2+v^2+2\Phi' v) \rightarrow$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi'\partial^\mu\Phi' + \Phi'\underbrace{\left(-\mu^2 v - \frac{\lambda}{4}2v^3 - \frac{\lambda}{4}2v^3\right)}_{\substack{-\mu^2 v - \lambda v^3 \\ +\lambda v^3 - \lambda v^3 = 0}} + \Phi'^2\underbrace{\left(-\frac{\mu^2}{2} - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}4v^2\right)}_{\substack{-\frac{\mu^2}{2} - \frac{\lambda}{2}v^2 - \lambda v^2 \\ \cancel{\frac{\lambda v^2}{2}} - \cancel{\frac{\lambda v^2}{2}} - \lambda v^2 \\ \mu^2}} +$$

$$\boxed{-\mu^2 = \lambda v^2}$$

$$+\Phi'^3\underbrace{\left(-\frac{\lambda}{4}2v - \frac{\lambda}{4}2v\right)}_{-\lambda v} - \frac{\lambda}{4}\Phi'^4$$

(76)

And the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + \mu^2\Phi'^2 - \lambda v\Phi'^3 - \frac{\lambda}{4}\Phi'^4$$

Note that now we have a real mass term because $\mu^2 < 0$ and the $\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi'$, $\mu^2\Phi'^2$ have opposite sign $\Rightarrow$ Klein-Gordon equation.

So the term $-\frac{\mu^2}{2}\Phi^2$ goes away and instead we get a $\mu^2\Phi'^2$ term which gives us

$$\frac{(\text{mass})^2}{2} = -\mu^2 > 0 \Rightarrow \text{mass} = \sqrt{-2\mu^2}$$

So the symmetry brake leads to massive boson field which does not exhibit the symmetry of the Lagrangian.

Another Model with Spontaneously Broken Symmetry

Consider now another example which has two fields: (77)

$$\mathcal{L}=\frac{1}{2}\left[\partial_\mu\sigma\partial^\mu\sigma+\partial_\mu\pi\partial^\mu\pi\right]-V(\sigma^2+\pi^2)$$

$$V(\sigma^2+\pi^2)=+\frac{1}{2}\mu^2(\sigma^2+\pi^2)+\frac{\lambda}{4}(\sigma^2+\pi^2)^2$$

The Lagrangian is invariant under $O(2)$, the group of 2-D rotations

$$\begin{pmatrix}\sigma'\\ \pi'\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix} \quad (A)$$

or if $\theta \ll$ $\begin{pmatrix}\sigma'\\ \pi'\end{pmatrix}\cong\begin{pmatrix}1 & \theta\\ -\theta & 1\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix}$

$$\cong\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}+\theta\underbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}_{\text{group generator}}\right\}\begin{pmatrix}\sigma\\ \pi\end{pmatrix}$$

$$\left.\begin{array}{l}\sigma'=\sigma+\theta\pi\\ \pi'=\pi-\theta\sigma\end{array}\right\}\Rightarrow \begin{array}{l}\pi\cong\frac{1}{1+\theta^2}(\pi'+\theta\sigma')\\ \sigma\cong\frac{1}{1+\theta^2}(\sigma'-\theta\pi')\end{array}$$

The potential depends on $\sigma^2+\pi^2$ only and using (A) it is easy to show that it is $O(2)$ invariant

(78)

$$(A)\rightarrow \begin{pmatrix}\sigma\\ \pi\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\sigma'\\ \pi'\end{pmatrix}$$

$$\sigma^2+\pi^2\rightarrow \cos^2\theta\,\sigma'^2+\sin^2\theta\,\pi'^2-\cancel{2\cos\theta\sin\theta\,\sigma'\pi'}+\sin^2\theta\,\sigma'^2+\cos^2\theta\,\pi'^2+\cancel{2\sin\theta\cos\theta\,\sigma'\pi'}$$

$$\rightarrow \sigma'^2+\pi'^2$$

$\therefore\ V(\sigma^2+\pi^2)$ = invariant under $O(2)$

$$\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma+\frac{1}{2}\partial_\mu\pi\partial^\mu\pi=\frac{1}{2}\partial_\mu(\cos\theta\sigma'-\sin\theta\pi')\partial^\mu(\cos\theta\sigma'-\sin\theta\pi')$$

$$+\frac{1}{2}\partial_\mu(\sin\theta\sigma'+\cos\theta\pi')\partial^\mu(\sin\theta\sigma'+\cos\theta\pi')$$

$$=\frac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma'+\frac{1}{2}\partial_\mu\pi'\partial^\mu\pi'$$

Therefore the Lagrangian is indeed invariant under $O(2)$ $[=U(1)]$

Continued……..

79

Just as we did before let $\mu^2 < 0$ $\lambda > 0$

$$V = \frac{1}{2}\mu^2(\sigma^2+\pi^2) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2$$

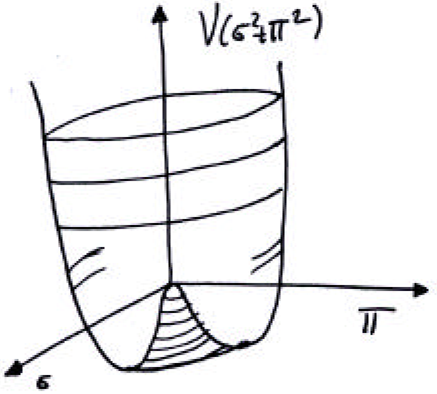
$$\frac{\partial V}{\partial \sigma} = \mu^2\sigma + \frac{\lambda}{4}2(\sigma^2+\pi^2)2\sigma = 0$$

$$= \sigma\left[\mu^2 + \lambda(\sigma^2+\pi^2)\right] = 0$$

$$\frac{\partial V}{\partial \pi} = \pi\left[\mu^2 + \lambda(\sigma^2+\pi^2)\right] = 0$$

So the potential has minimum for

$$\boxed{\sigma^2+\pi^2 = -\mu^2/\lambda}$$



80

Define again the radius as $v = \sqrt{-\mu^2/\lambda}$

$$\therefore\ v^2 = \sigma^2+\pi^2 = -\mu^2/\lambda$$

Transform as before $\quad s = \sigma - v = \sigma - \langle\sigma\rangle_0$

$$\pi = \pi$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{\mu^2}{2}(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu s\partial^\mu s + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{\mu^2}{2}(s^2+v^2+2sv+\pi^2) - \frac{\lambda}{4}(s^2+v^2+2sv+\pi^2)^2$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu s\partial^\mu s + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{\mu^2}{2}(s^2+v^2+2sv+\pi^2) - \frac{\lambda}{4}(s^2+v^2+2sv+\pi^2)(s^2+v^2+2sv+\pi^2)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu s\partial^\mu s + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + s^2\Big[-\frac{\mu^2}{2} - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}4v^2\Big] +$$

$$\left(\cancel{\frac{\lambda v^2}{2}} - \cancel{\frac{\lambda v^2}{2}} - \lambda v^2 = +\mu^2\right)$$

$$+\pi^2\Big[-\frac{\mu^2}{2} - \frac{\lambda}{4}v^2 - \frac{\lambda}{4}v^2\Big]$$

$$\left(\cancel{\lambda v^2/2} - \cancel{\lambda v^2/2} = 0\right)$$

$$+s\Big[-\frac{\mu^2}{2}2v - \frac{\lambda}{4}2v^3 - \frac{\lambda}{4}2v^3\Big]$$

$$\left(\cancel{\lambda v^3} - \cancel{\lambda v^3} = 0\right)$$

$$-\frac{\lambda}{4}\Big[2s^3v + s^2\pi^2 + 2s^3v + 2sv\pi^2 + \pi^2 s^2$$

$$+2sv\pi^2 + s^4 + \pi^4\Big]$$

The Goldstone Theorem

(81)

$$\mathcal{L}=\frac{1}{2}\partial_\mu s\,\partial^\mu s+\frac{1}{2}\partial_\mu\pi\,\partial^\mu\pi+\mu^2 s^2+0\cdot\pi^2+0\cdot s$$
$$-\frac{\lambda}{4}\left\{(s^2+\pi^2)^2+4s^3v+4sv\pi^2\right\}\to$$

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu s\,\partial^\mu s+\partial_\mu\pi\,\partial^\mu\pi\right)+\mu^2 s^2+0\,\pi^2+0\cdot s+$$
$$-\frac{\lambda}{4}(s^2+\pi^2)^2-\lambda v s(s^2+\pi^2)$$

Interesting stuff!!!

- By choosing $s=\sigma-v$, $\pi=\pi$ we made the choice that the vacuum of the theory is $\begin{pmatrix} v \\ 0 \end{pmatrix}$ (we have the freedom to choose a solution of $\sigma^2+\pi^2=-\mu^2/\lambda$)

  This vacuum breaks $O(2)$ (try it if you like)

- We started with a theory with two fields σ,π and no mass term and because the theory has a broken symmetry we got one field, the s, with mass and another field, the π, massless!!!

(82)

$$M_s^2=-2\mu^2>0 \quad (\text{because } \mu^2<0)$$
$$M_\pi^2=0$$

This is not an accident and comes from the GOLDSTONE THEOREM:

"If a theory has a continuous symmetry of the Lagrangian which is not a symmetry of the vacuum then we have a massless boson in the theory"

Consider a theory that has the $O(n)$ group. The Lagrangian of such a theory will look like

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i-\frac{\mu^2}{2}\phi^i\phi^i-\frac{\lambda}{4}(\phi^i\phi^i)^2$$

or if $\Phi=\begin{pmatrix}\phi_1\\ \vdots \\ \phi_n\end{pmatrix}$ $\quad \mathcal{L}=\frac{1}{2}\partial_\mu\tilde{\Phi}\,\partial^\mu\Phi-\frac{\mu^2}{2}\tilde{\Phi}\Phi-\frac{\lambda}{4}(\tilde{\Phi}\Phi)^2$

$\tilde{\Phi}=(\phi_1 \cdots \phi_n)$

The Goldstone Theorem II

Suppose again that $\mu^2 < 0$ and therefore (83)
there is a minimum of $\frac{\partial V}{\partial \phi_i} = 0$

$$\Phi_e\left[\mu^2 + \lambda(\phi^i\phi^i)\right] = 0$$

So as before one of the fields will develop a vacuum expectation value (VEV).

$$\langle\phi\rangle_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}$$

$O(n)$ has $\frac{1}{2}n(n-1)$ generators. Clearly not all are broken

And transformations like this:

$$\underbrace{\left[\begin{array}{c|c} O(n-1) & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{matrix} \\ \hline \times\times\times\;\times\;\times\times\times & 1 \end{array}\right]}_{A}$$

leave the vacuum invariant

$$A\langle\phi\rangle_0 = \langle\phi\rangle$$

So the vacuum is invariant under $O(n-1)$ but not under $O(n)$

The group $O(n-1)$ has $\frac{1}{2}(n-1)(n-2)$ (84)
generators, therefore the number of broken generators is

$$N(O(n)) - N(O(n-1)) =$$

$$\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$$

This you can see also from the previous picture: There are $n-1$ zeros which if they are replaced by something finite brake the symmetry (give another VEV)

So the most general form of our field is

$$\Phi = e^{\underbrace{i\xi_i \cdot K_i / v}_{n-1 \text{ broken generators}}} \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ m+v \end{pmatrix} \qquad 1 \leq i < n-1$$

Conclusion

Since complex numbers are involved, I will use "+" instead of "~" in the Lagrangian. So: (85)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^\dagger(\partial^\mu \phi) - \frac{\mu^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2$$

By substituting $\Phi = e^{i\xi_i \cdot k_i/v}\begin{pmatrix} 0 \\ \vdots \\ n+v \end{pmatrix}$ into the Lagrangian (Keep in mind the form of k_i) you get

$$\mathcal{L} = \frac{1}{2}(\partial_\mu n\, \partial^\mu n + \partial_\mu \xi^i \partial^\mu \xi^i) - \frac{\mu^2}{2}(n+v)^2 + - \frac{\lambda}{4}(n+v)^4 + \text{H.O.T}$$

- The n field again develops a VEV and the $n-1$ ξ_i remain massless!!!

Bottom line: THE NUMBER OF THE BROKEN GENERATORS IS EQUAL TO THE NUMBER OF MASSLESS GOLDSTONE BOSONS THAT EXIST IN THIS THEORY

Example: The Complex Scalar Lagrangian

We can demonstrate all these for n=2. Go back to the Lagrangian of the σ,π model and write the same Lagrangian in complex form: (86)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \mu^2/_2\,\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \;;\mu^2<0$$

Write $\Phi = \rho\, e^{i\theta/v}$ $\rho(x), \theta(x)$ are fields

$$v = \sqrt{\frac{-\mu^2}{\lambda}}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu(\rho e^{-i\theta/v})\partial^\mu(\rho e^{i\theta/v}) - \mu^2/_2\,\rho^2 - \frac{\lambda}{4}\rho^4$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\rho e^{-i\theta/v} - i\frac{\partial_\mu\theta}{v}\rho e^{-i\theta/v}\right)\left(\partial^\mu\rho e^{i\theta/v} + i\frac{\partial^\mu\theta}{v}\rho e^{i\theta/v}\right) +$$

$$-\frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\frac{1}{v^2}\partial_\mu\theta\partial^\mu\theta\,(\rho)^2 - \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

go to the minimum and expand $\rho = n + v$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho\partial^\mu\rho) + \underbrace{\frac{1}{2v^2}\partial_\mu\theta\partial^\mu\theta\,(n+v)^2}_{\underbrace{\frac{1}{2v^2}\partial_\mu\theta\partial^\mu\theta\,(n^2+v^2+2nv)}_{\frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \text{H.O.T}}} - \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

(87)

And the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\partial_\mu\theta\partial^\mu\theta - \frac{\mu^2}{2}(n^2+v^2+2nv) +$$
$$-\frac{\lambda}{4}(n^2+v^2+2nv)$$

Observations: ① No $\frac{m^2}{2}\theta^2$ term $\Rightarrow \boxed{m_\theta = 0}$ and θ is a Goldstone Boson

② The coefficient of n^2 is $\left[-\cancel{\frac{\mu^2}{2}} - \cancel{\frac{\lambda}{4}}v^2 - \frac{\lambda}{4}4v^2 - \cancel{\frac{\lambda}{4}}v^2\right]$

$$= \mu^2 < 0$$

$$\Rightarrow m^2_{ass}(n) = -2\mu^2$$

$$\boxed{m_n^2 = -2\mu^2 > 0}$$

∴ The theory has one Goldstone boson ($m_\theta = 0$) because one generator is broken and one massive scalar (boson).