Non-Abelian Symmetries

Non-Abelian Transformations

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Last time we worked out an example with the $U(1)$ group which is an abelian group. That is, the group operations commute: $e^{ia(x)} \cdot e^{ib(x)} = e^{ib(x)} \cdot e^{ia(x)}$

Today we will consider groups where the group operations do not commute

Consider for example the $SU(2)$ group. Under this group fields transform as

$$\Phi \to \Phi' = e^{-i\frac{\vec{\tau}}{2}\cdot\vec{\theta}}\,\Phi \quad (A)$$

where $\vec{\tau} = \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices and $\vec{\theta} = \vec{\theta}(x)$ is an x-dependent vector.

under (A) $\delta\Phi = -\frac{i}{2}(\vec{\tau}\cdot\vec{\theta})\cdot\Phi$
$= -\frac{i}{2}(\vec{\tau}\cdot\vec{\theta})_{ij}\,\Phi_j$

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Were Φ_j $j=1,2$ are two complex fields forming an $SU(2)$ doublet:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

The Lagrangian:

$$\mathcal{L} = \frac{1}{2}[\partial_\mu\phi]^\dagger\partial^\mu\phi - \frac{m^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2$$

has the $SU(2)$ global symmetry.

As before we would like to "gauge" this symmetry and make the Lagrangian invariant also under local $SU(2)$ $\{\vec{\theta} = \vec{\theta}(x)\}$.

As we did with $U(1)$ symmetry we require that $\partial_\mu \to D_\mu = \partial_\mu - ig\vec{\tau}\cdot\vec{A}_\mu$.

Note # of fields A^i_μ = # of group generators

In general: $SU(N) \Rightarrow N^2-1$ generators
$O(N) \Rightarrow \frac{N}{2}(N-1)$ generators

$\Longrightarrow$ $SU_c(3) \to 8$ generators, $SU(2) \to 3$ generators

Gauging a Non-Abelian Theory

This is the reason that the mediators of the weak interactions are three ($W^\pm, Z^0$) and also for having 8 different gluons in QCD. (66)

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To "gauge" the theory and make the Lagrangian invariant under SU(2) local transformations require as with U(1):

$$D'_\mu \Phi'(x) = D'_\mu U(\vec\theta)\Phi(x) = U(\vec\theta) D_\mu \Phi(x) \Rightarrow$$

if $D_\mu = \partial_\mu - ig\vec\tau\cdot\vec A_\mu$

$$(\partial'_\mu - ig\vec\tau\vec A'_\mu)U(\vec\theta)\Phi(x) = U(\vec\theta)(\partial_\mu - ig\vec\tau\cdot\vec A_\mu)\Phi$$

$$\partial_\mu U(\theta)\cdot\Phi + \cancel{U(\theta)\partial_\mu\Phi} - ig\vec\tau\vec A'_\mu U(\vec\theta)\Phi(x) =$$

$$\cancel{U(\theta)\partial_\mu\Phi} - ig\,U(\vec\theta)\vec\tau\cdot\vec A_\mu\Phi \Rightarrow$$

$$\partial_\mu U(\vec\theta) - ig\,\vec\tau\vec A'_\mu U(\vec\theta) = -ig\,U(\vec\theta)\vec\tau\cdot\vec A_\mu \Rightarrow$$

$$\partial_\mu U(\vec\theta)\cdot U^{-1}(\theta) - ig\,\vec\tau\cdot\vec A'_\mu = -ig\,U(\vec\theta)\vec\tau\cdot\vec A_\mu U^{-1}(\theta)$$

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$$\rightarrow ig\vec\tau\vec A'_\mu = ig\,U(\vec\theta)\vec\tau\cdot\vec A_\mu U^{-1}(\vec\theta) + \partial_\mu U(\vec\theta)U^{-1}(\vec\theta)$$

$$\boxed{\vec\tau\cdot\vec A'_\mu = U(\vec\theta)\vec\tau\cdot\vec A_\mu U^{-1}(\theta) - \frac{i}{g}\partial_\mu U(\vec\theta)U^{-1}(\vec\theta)}$$

Define $\mathbb{A}'_\mu = \vec\tau\cdot\vec A'_\mu = \tau^1 A^{1\prime}_\mu + \tau^2 A^{2\prime}_\mu + \tau^3 A^{3\prime}_\mu$

$\mathbb{A}_\mu = \vec\tau\cdot\vec A_\mu = \tau_1 A^1_\mu + \tau^2 A^2_\mu + \tau^3 A^3_\mu$

Then we have that:

$$\boxed{\mathbb{A}'_\mu = U(\theta)\mathbb{A}_\mu U^{-1}(\theta) - \frac{i}{g}\partial_\mu U(\vec\theta)\cdot U^{-1}(\theta)} \quad (G)$$

Summary:

If $\Phi \rightarrow \Phi' = U(\vec\theta)\Phi = e^{-i\frac{\vec\tau}{2}\cdot\vec\theta}\Phi$ and

$$\mathbb{A}'_\mu = U(\theta)\mathbb{A}_\mu U^{-1}(\vec\theta) - \frac{i}{g}\partial_\mu U(\theta)\cdot U^{-1}(\theta)$$

Then the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi)$$

is invariant under local gauge transf. based on the SU(2) group.

The Lagranian for a Non-Abelian Gauge Field

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But the kinetic term for the gauge fields is still missing:

Define $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\,\epsilon^{ijk} A^j_\mu A^k_\nu$

and $\mathbb{F}_{\mu\nu} = T^i F^i_{\mu\nu}$

Then as you will show for <u>Homework II</u>

the Lagrangian $\mathcal{L} = -\frac{1}{4} Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})$

is invariant under (G).

Therefore the full Lagrangian of the theory is:

$$\boxed{\begin{gathered}\mathcal{L} = \mathcal{L}_{GAUGE} + \mathcal{L}_{Scalar} \rightarrow: \\ \mathcal{L} = -\frac{1}{4} Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}) + \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^\dagger\phi)\end{gathered}}$$

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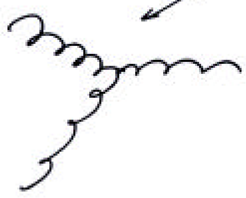
Note: (a) $[T^i\, T^j] = i\,\epsilon^{ijk}\,T^k$

$Tr(T^i T^j) = \kappa \cdot \delta^{ij}$ κ=constant

(b) $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \underbrace{g\,\epsilon^{ijk} A^j_\mu A^k_\nu}$



The $g\,\epsilon^{ijk} A^j_\mu A^k_\nu$ term which appears in non-abelian theories is responsible for these couplings which do not exist in abelian theories.

Non-Abelian Gauge Transformations

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$$A'_\mu = U A_\mu U^{-1} - \frac{i}{g}\partial_\mu U \cdot U^{-1}$$

$$U(\vec{\theta}) = e^{-i\vec{\tau}\cdot\vec{\theta}/2} = 1 - \frac{i}{2}\vec{\theta}\cdot\vec{\tau} + \cdots$$

$$T^i A^{i\prime}_\mu = \left(1 - i/_2\,\vec{\theta}\cdot\vec{\tau}\right) T^\ell A^\ell_\mu \left(1 + \frac{i}{2}\vec{\theta}\cdot\vec{\tau}\right) + \\ - i/g\left(-\frac{i}{2}\partial_\mu\vec{\theta}\cdot\vec{\tau}\right)\cancel{UU^{-1}} \rightarrow$$

$$T^i A^{i\prime}_\mu = \left(T^\ell A^\ell_\mu - i/_2\,\theta^k T^k T^\ell A^\ell_\mu\right)\left(1 + i\frac{\theta^j T^j}{2}\right) \\ - i/g\left(-i/_2\,\partial_\mu\vec{\theta}\cdot\vec{\tau}\right) \rightarrow$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu - \frac{i}{2}T^k T^\ell \theta^k A^\ell_\mu + \frac{i}{2}T^\ell T^j \theta^j A^\ell_\mu \\ - i/g\left(-i/_2\,\partial_\mu\theta^i T^i\right)$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu + \frac{i}{2}T^\ell T^k \theta^k A^\ell_\mu - \frac{i}{2}T^k T^\ell \theta^k A^\ell_\mu \\ - \frac{1}{2g}\partial_\mu\theta^i T^i$$

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$$\rightarrow T^i A^{i\prime}_\mu = T^i A^i_\mu + \frac{i}{2}[T^\ell, T^k]\theta^k A^\ell_\mu + \\ - \frac{1}{2g}\partial_\mu\theta^i T^i \rightarrow$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu + \frac{i}{2}\, i\,\epsilon^{\ell k m} T^m \theta^k A^\ell_\mu + \\ - 1/2g\,\partial_\mu\theta^i T^i$$

$$T^i A^{i\prime}_\mu = T^i A^i_\mu - \frac{1}{2}\epsilon^{m\ell k} T^m \theta^k A^\ell_\mu \\ - 1/2g\,\partial_\mu\theta^i T^i$$

$$A^{i\prime}_\mu = A^i_\mu - \frac{1}{2}\epsilon^{i\ell k}\theta^k A^\ell_\mu - \frac{1}{2g}\partial_\mu\theta^i$$

$$A^{i\prime}_\mu = A^i_\mu + \frac{1}{2}\epsilon^{ik\ell}\theta^k A^\ell_\mu - \frac{1}{2g}\partial_\mu\theta^i$$

this would be the non-abelian version of $A'_\mu = A_\mu + \partial_\mu\theta$ in QED

Exercises

Exercise: Calculate $[\mathbb{D}_\mu, \mathbb{D}_\nu]\Phi$ (#2)

Hint: you should be getting the result in terms of $\mathbb{F}_{\mu\nu}$

Exercise: Show that:

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu - ig[\mathbb{A}_\mu, \mathbb{A}_\nu]$$