Global and Local Abelian Symmetries

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GLOBAL AND LOCAL SYMMETRIES
OF THE LAGRANGIAN

Consider the tranformation where:

$$\Phi \rightarrow e^{ie}\Phi \quad \text{where } \underline{e \text{ is constant.}}$$

The Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^*\partial^\mu\Phi - \frac{m^2}{2}\Phi^*\Phi$ is invariant under this transformation and using Noether's theorem we have that

$$J^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\delta\Phi_i$$

(Φ, Φ^* are two independent degrees of freedom. If you prefer $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$)

$$\left.\begin{array}{l} J^\mu = \dfrac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i^*}\delta\Phi_i^* + \dfrac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\delta\Phi_i \\ \\ \delta\Phi = ie\Phi \end{array}\right\} \Rightarrow$$

$$J^\mu = \frac{1}{2}\partial^\mu\Phi(-ie\Phi^*) + \frac{1}{2}\partial^\mu\Phi^*(ie\Phi) \rightarrow$$

$$J^\mu = \frac{ie}{2}\left(\Phi\partial^\mu\Phi^* - \Phi^*\partial^\mu\Phi\right)$$

and by Noether's theorem $\partial_\mu J^\mu = 0$

$\therefore$ An internal global symmetry (continuous) leads to a conserved current.

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This kind of transformations are called $\underline{\text{global}}$ transformations because the transformation has e^{ie} where e is a constant which is the same everywhere in space-time. One could have a constant matrix then which would be the generator of an $SU(n)$ group.

$e^{ie} \rightarrow U(1)$ group.

One can extend this concept by letting the exponent to be space time dependent. Then we have the $\underline{\text{local transformations}}$:

$$\text{Local } U(1): \quad \Phi(x) \rightarrow \Phi'(x) = e^{ie\alpha(x)}\Phi(x)$$

(note that $\alpha(x)$ is space-time dep.)

<u>$U(1)$ abelian group:</u>

1. $1 = e^{i0}$
2. $\forall\, g_i = e^{i\theta_i} \Rightarrow \exists\, e^{-i\theta_i} = g^{-1}: \; g \cdot g^{-1} = g^{-1} g = 1$
3. $g_i g_j = e^{i\theta_i}e^{i\theta_j} = e^{i(\theta_i + \theta_j)} = g_j g_i$
4. if $g_i \cdot g_j = g_e \Rightarrow g_e$ belongs to the $U(1)$ group

Local Symmetries

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Demand that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\nu\Phi^*\partial^\mu\Phi - \frac{m^2}{2}\Phi^*\Phi$$

is invariant under $\Phi \rightarrow e^{i\alpha(x)\cdot e}\Phi(x) = \Phi'(x)$

That is: The Lagrangian density is invariant under U(1) LOCAL GAUGE TRANSFORMATIONS. (We will see later why "GAUGE").

$$\Phi^*\Phi \rightarrow \Phi^{*\prime}\Phi' = e^{-ie\alpha(x)}\Phi^* e^{ie\alpha(x)}\Phi = \Phi^*\Phi$$

$\Rightarrow$ INVARIANT

$$\partial_\mu\Phi \rightarrow \partial_\mu\Phi' = \partial_\mu\left\{e^{i\alpha(x)\cdot e}\Phi(x)\right\}$$

$$\rightarrow e^{i\alpha(x)\cdot e}\partial_\mu\Phi(x) + ie\partial_\mu\alpha(x)e^{i\alpha(x)e}\Phi(x)$$

$$\rightarrow \underbrace{e^{i\alpha(x)e}\partial_\mu\Phi(x)}_{\text{ok term the phase will go away when } \partial_\mu\Phi^*\partial^\mu\Phi} + \underbrace{e^{i\alpha(x)e}\Phi(x)(-ie\partial_\mu\alpha(x))}_{\text{No good, depends upon } \alpha(x) \text{ and this will not go away...}}$$

Clearly if we want $\mathcal{L}$ invariant we must change something to fix this problem.

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The only reasonable option we have is to change $\partial_\mu \rightarrow D_\mu$ (yet to be defined) which somehow will kill the $\partial_\mu\alpha(x)$ terms.

Introduce a vector field A_μ (∂_μ is a vector) such that $D_\mu = \partial_\mu - ieA_\mu(x)$

The idea is that while $\Phi \rightarrow \Phi'$ $A_\mu(x)$ also $A_\mu \rightarrow A_\mu'$ such that $\mathcal{L}$ remain invariant.

In math:

$$(\partial_\mu - ieA_\mu')\underbrace{U\Phi(x)}_{\Phi'(x)} = U(\partial_\mu - ieA_\mu)\Phi(x)$$

$$\partial_\mu\left(e^{ie\alpha(x)}\Phi(x)\right) - ieA_\mu' e^{ie\alpha(x)}\Phi(x) = e^{ie\alpha(x)}\partial_\mu\Phi(x) - ieA_\mu\Phi(x)e^{ie\alpha(x)} \rightarrow$$

$$ie\partial_\mu\alpha(x)\Phi(x)e^{ie\alpha(x)} + \cancel{e^{ie\alpha(x)}\partial_\mu\Phi(x)} - ieA_\mu' e^{ie\alpha(x)}\Phi(x) = \cancel{e^{ie\alpha(x)}\partial_\mu\Phi(x)} - ieA_\mu\Phi(x)e^{ie\alpha(x)} \rightarrow$$

$$\cancel{ie}\partial_\mu\alpha(x) - \cancel{ie}A_\mu' = -\cancel{ie}A_\mu \rightarrow$$

$$\boxed{A_\mu' = A_\mu + \partial_\mu\alpha(x)}$$

Summary and Conclusions

Summary: To make $\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2}{2}\phi^*\phi$ (57)
invariant under U(1) where

$$\Phi \to \Phi'(x) = e^{ie\alpha(x)}\Phi(x)$$

we introduce

① $D_\mu = \partial_\mu - ieA_\mu$, D_μ is called COVARIANT DERIVATIVE and A_μ is a spin 1 (Vector) field

② $A'_\mu = A_\mu + \underbrace{\partial_\mu\alpha(x)}_{\text{gauge}}$

One can immediately see that:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

is invariant under local U(1) transformations

It is also obvious that a term

$$m^2 A_\mu A^\mu$$

which would give mass to the field A^μ is prohibited by local gauge invariance

In conclusion the full Lagrangian of this theory can be written as: (58)

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{m^2}{2}\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This theory has the U(1) local gauge symmetry. The "gauge" term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ gives us the Maxwell Eq. Therefore it describes the vector field A_μ called photon.

The first <u>prediction</u> of our U(1) theory is that the photon mass $m_\gamma = 0$ otherwise U(1) local symmetry breaks down.

The first and the second terms describe a spin zero (scalar) particle with 2 DoF and mass m.

But our demand to have $\mathcal{L}(x)$ with local gauge invariance leads to more predictions:

Scalar Lagrangian with local U(1) Symmetry

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$$\mathcal{L}_{s=0} = \frac{1}{2}(\partial_\mu + ieA_\mu)\Phi^*(\partial^\mu - ieA^\mu)\Phi - \frac{m^2}{2}\phi^*\phi$$

$$\mathcal{L}_{s=0} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2}{2}\phi^*\phi + \frac{1}{2}ieA_\mu\Phi^*\partial^\mu\phi +$$

$$-\frac{i}{2}eA^\mu\partial_\mu\phi^*\cdot\phi + \frac{1}{2}(-ie)(+ie)A_\mu A^\mu\phi^*\phi$$

$$\mathcal{L}_{s=0} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2}{2}\phi^*\phi + \underbrace{\frac{ie}{2}(\Phi^*\partial^\mu\phi - \Phi\partial^\mu\phi^*)A_\mu}_{J_\mu A^\mu}$$

$$+\frac{e^2}{2}A_\mu A^\mu\Phi^*\Phi$$

As you should by now recognise from our field theory introduction $H_I = J_\mu \cdot A^\mu + \frac{e^2}{2}A_\mu A^\mu \Phi^*\Phi$

and the transition amplitude will have a term $e^{i\int d^4x\,(\underbrace{J_\mu A^\mu + e^2/2\, A_\mu A^\mu \phi^*\phi}_{H_I})}$

So by expanding this one gets

$$1 + i\int d^4x\left[J_\mu A^\mu + \frac{e^2}{2}A_\mu A^\mu\phi^*\phi\right] + .$$

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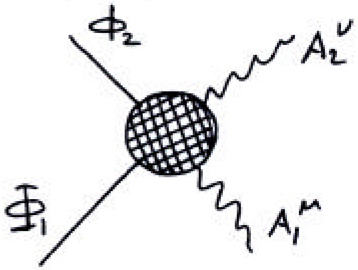
It appears that our Lagrangian describes the Electrodynamics of massive Scalar fields. Or put in a different way: It describes the interactions of a massless vector field, the photon, with massive scalar and charged fields. This can be shown as follows:

Consider the interaction of a photon with the charged scalar particle:



As we have already learned in Field theory the Scattering amplitude will be:

$$A = \langle 0|\,T\left\{\Phi_2\Phi_1 A_1^\mu A_2^\nu\left(1 + i\int d^4x\left[J_\rho A^\rho + \frac{e^2}{2}A_\rho A^\rho\phi^*\phi + \ldots\right]\right)\right\}|0\rangle$$

The e^0 term gives disconnected diagrams:

ϕ_2, A_2^ν, ϕ_1, A_1^μ which have zero amplitude

The Scalar QED Feynman Diagrams

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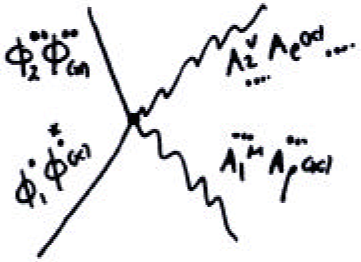
The e^2 term will give :

$$\langle 0|T(\phi_1\phi_2 A_1^\mu A_2^\nu (\tfrac{i}{2}e^2 A_\rho(x) A^\rho(x) \Phi^*(x)\Phi^*(x))|0\rangle$$

$$\Phi_1^{\circ}\, \overset{*\circ}{\Phi}(x)\, \phi_2^{\circ\circ}\, \overset{*\circ\circ}{\Phi}(x)\, A_1^{\overset{\circ\circ\circ}{\mu}}\, A_\rho(x)\, A_2^{\overset{\circ\circ\circ\circ}{\nu}}\, A^\rho(x)$$



So our theory predicts a "point like" Compton scattering with strength e^2 and if you carry the full calculation you will find that the Feynman rule for this diagram is

$$2ie^2 g_{\mu\nu}$$

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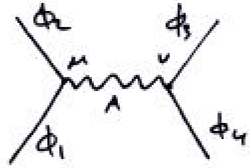
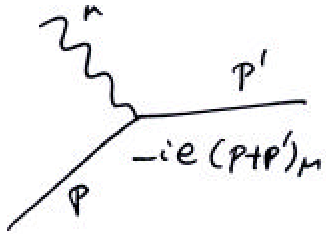
In a similar way the $J_\mu A^\mu$ term which is order-e will result to e^2 terms corresponding to diagrams that look like:



order e^2

and correspond to the Feynman rule



So the amplitude to first order perturbation will go like e^2 just like in QED but the Feynman rules are different

Conclusions

Conclusion

1. Consider: $\mathcal{L} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - \frac{m^2}{2}\phi^*\phi$

   It describes a scalar (s=0) charged field.

2. Demand that $\mathcal{L}$ is invariant under

$$\Phi \to \Phi' = e^{ie\alpha(x)}\Phi(x) \quad U(1) \text{ group}$$

3. To do this you have to introduce:

$$A_\mu , \; D_\mu = \partial_\mu - ieA_\mu$$

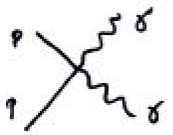
and $$A'_\mu = A_\mu + \partial_\mu \alpha(x)$$

4.- The Lagrangian becomes:

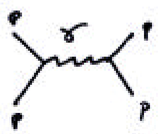
$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{m^2}{2}\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

5.- It predicts: (a) $m_\gamma = 0$

(b)



(c)



And all these by just requiring that the Lagrangian is invariant under LOCAL GAUGE TRANS.