Feynman Diagrams

- Start with Coulomb Scattering just to show the different steps in calculating a cross-section, integrating over the entire phase space and summing over spins.
- Introduce the Feynman rules in QED.
- Calculate a cross section using the Feynman rules.
- Give you a Feynman Diagram to calculate.

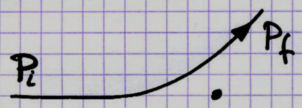
Cross-Section Calculations: Coulomb Scattering

(I)

COULOMB SCATTERING OF SPIN $\frac{1}{2}$ PARTICLES

21/11/05

This is going to be the first example of cross-section calculation



$$\psi_f = \sqrt{\frac{m}{E_f V}}\, u(p_f s_f)\, e^{-ip_f x}$$

$$\psi_i = \sqrt{\frac{m}{E_i V}}\, u(p_i s_i)\, e^{-ip_i x}$$

$$A^\mu = (A^0, \vec{0}) = \left(-\frac{Ze}{4\pi r}, \vec{0}\right)$$

V is the volume element

$$S_{fi} = -ie \int d^4x\, \overline{\psi}_f A_\mu(x) \gamma^\mu \psi_i \quad f \neq i \Rightarrow$$

$$S_{fi} = +i \frac{Ze^2}{4\pi} \frac{m}{V} \frac{1}{\sqrt{E_i E_f}} \int d^4x \frac{\overline{u}(p_f s_f) \gamma^0 u(p_i s_i)}{|\vec{x}|} \times e^{i(p_f - p_i)\cdot x}$$

$$S_{fi} = +i \frac{Ze^2}{4\pi} \frac{m}{V} \frac{1}{\sqrt{E_i E_f}} \int dx^0\, e^{+i(E_f - E_i)x^0} \times$$

$$\times \int d^3\vec{x}\, \frac{e^{-i(\vec{p}_f - \vec{p}_i)\cdot\vec{x}}}{|\vec{x}|}\, \overline{u}(p_f s_f) \gamma^0 u(p_i s_i) \quad (A)$$

Next calculate the integral

$$I_1 = \int d^3\vec{x}\, \frac{e^{-i(\vec{p}_f - \vec{p}_i)\cdot\vec{x}}}{|\vec{x}|}$$

(II)

$$I_1 = \int d^3\vec{x}\, \frac{e^{-i(\vec{p}_f - \vec{p}_i)\cdot\vec{x}}}{|\vec{x}|} = \int d^3\vec{x}\, \frac{e^{-i\vec{Q}\cdot\vec{x}}}{|\vec{x}|} \Rightarrow$$

$$\vec{Q} = \vec{p}_f - \vec{p}_i$$

$$I_1 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^\infty r^2 dr \frac{1}{r} e^{-i|\vec{Q}| r\cos\theta} \rightarrow$$

$$I_1 = 2\pi \int_1^{-1} (-1)\, d\cos\theta \int_0^\infty r\, dr\, e^{-i|\vec{Q}| r \cos\theta} \Rightarrow$$

$$I_1 = 2\pi \int_0^\infty r\, dr \int_{-1}^{1} d\cos\theta\, e^{-i|\vec{Q}| r\cos\theta} \rightarrow$$

$$I_1 = 2\pi \int_0^\infty \cancel{r}\, dr \frac{1}{(-i)|\vec{Q}|\cancel{r}} \left(e^{-i|\vec{Q}|r} - e^{+i|\vec{Q}|r}\right) \rightarrow$$

$$I_1 = 2\pi \int_0^\infty dr \frac{1}{\cancel{(-i)}|\vec{Q}|}\, \cancel{2i(-1)} \sin(|\vec{Q}| r) \Rightarrow$$

$$\left.\begin{array}{l} I_1 = \dfrac{4\pi}{|\vec{Q}|} \displaystyle\int_0^\infty dr \sin(|\vec{Q}| r) = \dfrac{4\pi}{|\vec{Q}|} \int_0^\infty dr\, \mathrm{Im}\, e^{iQr} \\ Q \rightarrow Q + i\mu \quad \mu > 0 \\ \text{at the end } \mu \rightarrow 0 \end{array}\right\} \Rightarrow$$

$$I_1 = \frac{4\pi}{|\vec{Q}|} \mathrm{Im} \lim_{\mu\to 0} \underbrace{\int_0^\infty dr\, e^{+i(Q+i\mu)r}}_{\frac{1}{i(Q+i\mu)}(0-1)}$$

$$\frac{-1}{-\mu + iQ} = \frac{-1(-\mu - iQ)}{\mu^2 + Q^2} = \frac{\mu + iQ}{\mu^2 + Q^2}$$

$$I_1 = \frac{4\pi}{|\vec{Q}|} \mathrm{Im}\left(\lim_{\mu\to 0} \frac{\mu + iQ}{\mu^2 + Q^2}\right) = \frac{4\pi}{|\vec{Q}|} \mathrm{Im}\left(\frac{i}{|\vec{Q}|}\right) \Rightarrow \boxed{I_1 = \frac{4\pi}{|\vec{Q}|^2}}$$

Wave packets vs. plane waves

III

at the end the amplitude can be written as:

$$S_{fi} = i\frac{Ze^2}{4\pi}\frac{m}{V}\frac{1}{\sqrt{E_iE_f}}\frac{4\pi}{|\vec{Q}|^2}\bar{U}(P_fS_f)\gamma^0U(P_iS_i)\times$$

$$\times\int dx^0 e^{+i(E_f-E_i)X^0} \qquad (\alpha)$$

if you now replace the integral with $2\pi\delta(E_f-E_i)$ as you should, you get a problem when you square to compute the probability. The reason for this infinity is that we have not used wave packets but plane waves which means we incorrectly have to integrate from $-\infty < x^0 < +\infty$. Fermi comes to help with his golden rule:

Lets calculate the integral for finite time

$$\int_0^t dt\, e^{i(E_f-E_i)t} = \frac{1}{i(E_f-E_i)}\left(e^{i(E_f-E_i)t}-1\right)$$

$$= \frac{1}{i(E_f-E_i)}e^{i(E_f-E_i)t/2}\left(e^{i(E_f-E_i)t/2}-e^{-i(E_f-E_i)t/2}\right)$$

$$= \frac{1}{\cancel{i}(E_f-E_i)}e^{i(E_f-E_i)t/2}\,2\cancel{i}\,\sin\left[\frac{(E_f-E_i)t}{2}\right]$$

$$= \frac{2}{(E_f-E_i)}e^{i(E_f-E_i)t/2}\sin\left(\frac{(E_f-E_i)t}{2}\right) \qquad (\beta)$$

IV

$(\alpha)\ (\beta) \rightarrow$

$$S_{fi} = i\frac{Ze^2}{4\pi}\frac{m}{V}\frac{1}{\sqrt{E_iE_f}}\frac{4\pi}{|\vec{Q}|^2}\bar{U}(P_fS_f)\gamma^0U(P_iS_i)$$

$$\times\frac{2}{(E_f-E_i)}e^{i(E_f-E_i)t/2}\sin\left[\frac{(E_f-E_i)t}{2}\right]$$

$$P_{fi} = |S_{fi}|^2 \Rightarrow$$

$$P_{fi} = \left(\frac{Ze^2}{4\pi}\right)^2\frac{m^2}{V^2}\frac{1}{E_fE_i}\left(\frac{4\pi}{|\vec{Q}|^2}\right)^2\times$$

$$\times\left|U(P_fS_f)\gamma^0U(P_iS_i)\right|^2\times$$

$$\frac{4}{(E_f-E_i)^2}\sin^2\left[\frac{(E_f-E_i)}{2}t\right] \qquad (\delta)$$

Now concentrate on

$\frac{4}{(E_f-E_i)^2}\sin^2\left(\frac{(E_f-E_i)t}{2}\right)$ which gives a contribution summed overall E_f which is

$$4\rho(E_f)\int_{-\infty}^{+\infty}\frac{\sin^2\left(\frac{E_f-E_i}{2}t\right)}{(E_f-E_i)^2}dE_f = 4\rho(E_f)\frac{2t^2}{t\,4}\int_{-\infty}^{+\infty}\frac{\sin^2\left(\frac{E_f-E_i}{2}t\right)}{\left(\frac{E_f-E_i}{2}t\right)^2}\times d\left(\frac{E_f-E_i}{2}t\right)$$

$$= 2\rho(E_f)t\underbrace{\int_{-\infty}^{+\infty}\frac{\sin^2\phi}{\phi^2}}_{\pi} = 2\pi t\rho(E_f)$$

Fermi's Golden Rule

V

This is what Fermi stated by his golden rule: The time integral contribution when calculating propabilities is

$$2\pi t \rho(E_f)$$

in other words $P_{fi} = 2\pi t \, |\langle f|H_{INT}|i\rangle|^2 \rho(E_f)$

and $\Gamma_{fi} = \frac{dP_{fi}}{dt} = 2\pi \, |\langle f|H_{INT}|i\rangle|^2 \rho(E_f)$

Γ_{fi} is the rate of interaction, which if divided by the incident flux of particles and integrated over the entire final state will give you the cross-section

Back to Coulomb scattering now:

$$P_{fi} = \left(\frac{Ze^2}{\cancel{4\pi}}\right)^2 \frac{m^2}{V^2} \frac{1}{E_f E_i} \left(\frac{\cancel{4\pi}}{|\vec{Q}|^2}\right)^2 \times$$

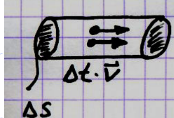
$$\times \, |\bar{u}(p_f s_f)\gamma^0 u(p_i s_i)|^2 \times$$

$$\underbrace{2\pi t \, \rho(E_f)}_{\delta(E_f - E_i)}$$

$$\Gamma_{fi} = \frac{dP_{fi}}{dt} = \frac{Z^2 e^4}{|\vec{Q}|^4} \frac{m^2}{V^2} \frac{1}{E_f E_i} |\bar{u}(p_f s_i)\gamma^0 u(p_i s_i)|^2 \, 2\pi \times \delta(E_f - E_i)$$

VI

The cross-section can now be computed using Γ_{fi} and the flux $\vec{F}$ of the incident beam.



$$\vec{F} = \frac{d(x) \cdot \Delta S \cdot \Delta t \cdot \vec{V}}{\Delta S \cdot \Delta t} = d(x)\,\vec{V} \sim cm^{-2}\,sec^{-1}$$

$d(x)$ = particle density in the beam

$$\Rightarrow \quad d\sigma = \frac{\Gamma_{fi}}{|\vec{F}|} = \frac{\Gamma_{fi}}{d \cdot |\vec{V}|} = \frac{\Gamma_{fi}}{|\vec{V}_i|} V$$

NORMALIZED TO 1 PARTICLE PER V, JUST LIKE THE SPINORS WERE

$$d\sigma = \frac{Z^2 e^4}{|\vec{Q}|^4} \frac{m^2}{|\vec{V}_i| \cdot V^2} \cancel{V} \frac{1}{E_f E_i} |\bar{u}_f \gamma^0 u_i|^2 \times 2\pi\,\delta(E_f - E_i) \times \underbrace{\frac{V d^3 p_f}{(2\pi)^3}}_{\text{PHASE SPACE}}$$

In general from now on we start from

$$d\sigma = \underbrace{\left(\frac{V}{|\vec{V}|}\right)}_{\text{FLUX NORMALIZATION}} \underbrace{2\pi\,|\langle f|H_{int}|i\rangle|^2 \rho(E_f)}_{\text{TIME INTEGRAL TAKEN CARE BUT NOT THE SPACE ONE}} \times \frac{V d^3 p_f}{(2\pi)^3}$$

Back to work:

$$d\sigma = \frac{Z^2 e^4}{|\vec{Q}|^4} \frac{m^2}{|\vec{V}_i|} \frac{1}{E_i E_f} |u(p_f s_f)\gamma^0 u(p_i s_i)|^2 \, 2\pi\,\delta(E_f - E_i) \times \frac{d^3 p_f}{(2\pi)^3}$$

Summing over spins

VII

Note that $\alpha_{QED} = \alpha = \frac{e^2}{4\pi} = \frac{1}{137}$

$$\therefore d\sigma = \frac{Z^2(\alpha\, 4\pi)^2}{|\vec{Q}|^4} \frac{m^2}{V_i E_f E_i} |u_f \gamma^0 u_i|^2 2\pi\delta(E_f - E_i) \frac{d^3 p_f}{(2\pi)^3}$$

$$d\sigma = \frac{4Z^2\alpha^2}{|\vec{Q}|^4} \frac{m^2}{V_i E_i E_f} |u_f \gamma^0 u_i|^2 \delta(E_f - E_i) \underbrace{d^3 p_f}_{p_f^2 dp_f d\Omega}$$

$$\frac{d\sigma}{d\Omega} = \int \frac{4Z^2\alpha^2}{|\vec{Q}|^4} \frac{m^2}{V_i E_i E_f} |u_f \gamma^0 u_i|^2 \delta(E_f - E_i) p_f^2 dp_f$$

but $E_f^2 = p_f^2 + m^2 \rightarrow \cancel{2} E_f dE_f = \cancel{2} p_f dp_f \Rightarrow$

$$\boxed{dp_f = \frac{E_f dE_f}{p_f}}$$

So

$$\frac{p_f^2 dp_f}{|\vec{V}_i| E_i E_f} = \frac{p_f^{\cancel{2}}}{|\vec{V}_i| E_i \cancel{E_f}} \frac{\cancel{E_f} dE_f}{\cancel{p_f}} = \frac{p_f dE_f}{|\vec{V}_i| E_i} = \frac{p_f dE_f}{\frac{p_i}{\cancel{E_i}} \cancel{E_i}}$$

$$= \frac{p_f dE_f}{p_i}$$

$$\therefore d\sigma = \int \frac{4Z^2\alpha^2}{|\vec{Q}|^4} m^2 |u_f \gamma^0 u_i|^2 \delta(E_f - E_i) \frac{p_f}{p_i} dE_f$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2 m^2}{|\vec{Q}|^4} |u(p_f s_f) \gamma^0 u(p_i s_i)|^2$$

VIII

Assume that the original beam is unpolarized so we need to average over the initial spins: this means "$\frac{1}{2}\sum_{s_i}$". We also need to sum over the final spins "$\sum_{s_f}$"

$$\frac{d\sigma^{TOT}}{d\Omega} = \frac{4Z^2\alpha^2 m^2}{|\vec{Q}|} \frac{1}{2} \sum_{s_i} \sum_{s_f} |u(p_f s_f) \gamma^0 u(p_i s_i)|^2 \qquad (A_1)$$

Consider now the spin sum: $A = \sum_{s_i s_f} |u_f \gamma^0 u_i|^2$

$$A = \sum_{s_f s_i} \bar{u}_\alpha(p_f s_f) \gamma^0_{\alpha\beta} u_\beta(p_i s_i) \left(\bar{u}(p_f s_f) \gamma^0 u(p_i s_i)\right)^\dagger$$

$$\underbrace{\underbrace{u^\dagger(p_i s_i) \gamma^0 (u^\dagger(p_f s_f) \gamma^0)^\dagger}_{\bar{u}(p_i s_i) \gamma^0 u(p_f s_f)}}_{\bar{u}_\lambda(p_i s_i) \gamma^0_{\lambda\nu} u_\nu(p_f s_f)}$$

$$A = \sum_{s_f s_i} \bar{u}_\alpha(p_f s_f) \gamma^0_{\alpha\beta} u_\beta(p_i s_i) \bar{u}_\lambda(p_i s_i) \gamma^0_{\lambda\nu} u_\nu(p_f s_f)$$

$$A = \sum_{s_i} \gamma^0_{\alpha\beta} \underbrace{u_\beta(p_i s_i) \bar{u}_\lambda(p_i s_i)}_{\left(\frac{\not{p}_i + m}{2m}\right)_{\beta\lambda}} \gamma^0_{\lambda\nu} \underbrace{\sum_{s_f} u_\nu(p_f s_f) \bar{u}_\alpha(p_f s_f)}_{\left(\frac{\not{p}_f + m}{2m}\right)_{\nu\alpha}}$$

$$A = \gamma^0_{\alpha\beta} \left(\frac{\not{p}_i + m}{2m}\right)_{\beta\lambda} \gamma^0_{\lambda\nu} \left(\frac{\not{p}_f + m}{2m}\right)_{\nu\alpha}$$

Two trace theorems

IX

IN CONCLUSION:

$$\sum_{S_i}\sum_{S_f} \left|\bar{u}(p_f S_f)\gamma^0 u(p_i S_i)\right|^2 = Tr\left(\gamma^0 \frac{\not p_i + m}{2m}\gamma^0 \frac{\not p_f + m}{2m}\right)$$

and the cross-section is then

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2 m^2}{2|\vec{Q}|^4}\, Tr\left(\gamma^0 \frac{\not p_i + m}{2m}\gamma^0 \frac{\not p_f + m}{2m}\right) \quad (T)$$

$$Tr\left(\gamma^0 \frac{\not p_i + m}{2m}\gamma^0 \frac{\not p_f + m}{2m}\right) = \frac{1}{4m^2} Tr\left(\gamma^0(\not p_i + m)\gamma^0(\not p_f + m)\right)$$

$$= \frac{1}{4m^2} Tr\left[(\gamma^0 \not p_i + m\gamma^0)(\gamma^0 \not p_f + \gamma^0 m)\right]$$

$$= \frac{1}{4m^2} Tr\left[\gamma^0 \not p_i \gamma^0 \not p_f + \underbrace{\gamma^0 \not p_i \gamma^0 m}_{0} + \underbrace{m\gamma^0\gamma^0 \not p_f}_{0} + m^2\gamma^0\gamma^0\right]$$

$$\boxed{\text{THE TRACE OF AN ODD \# OF } \gamma \text{ matrices is ZERO}}$$

$$= \frac{1}{4m^2} Tr\left[\gamma^0 \not p_i \gamma^0 \not p_f + m^2\right] = \frac{1}{4m^2}\left[Tr(\gamma^0 \not p_i \gamma^0 \not p_f) + 4m^2\right]$$

BUT

$$\boxed{Tr(\not a_1 \not a_2 \not a_3 \not a_4) = 4\left((a_1 \cdot a_2)(a_3 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3) - (a_1 \cdot a_3)(a_2 \cdot a_4)\right)}$$

$$= \frac{1}{4m^2}\left\{4(E_i E_f + E_f E_i - 1(E_i E_f - \vec{p}_i \cdot \vec{p}_f)) + 4m^2\right\}$$

X

$$= \frac{1}{4m^2} 4\left[2E_f E_i - E_f E_i + \vec{p}_f \cdot \vec{p}_i + m^2\right]$$

but $E_i = E_f$

$$= \frac{1}{m^2}\left[\cancel{2}E^2 - \cancel{E^2} + \vec{p}_f \cdot \vec{p}_i + m^2\right]$$

$$= \frac{1}{m^2}\left[E^2 + p^2\cos\theta + E^2 - \vec{p}^2\right] = \frac{1}{m^2}\left[2E^2 - p^2(1-\cos\theta)\right]$$

$$= \frac{1}{m^2}\left[2E^2 - 2p^2\sin^2\frac{\theta}{2}\right] = \frac{2}{m^2}\left(E^2 - p^2\sin^2\frac{\theta}{2}\right)$$

$$= \frac{2}{m^2}\left(E^2 - (\beta E)^2\sin^2\theta/2\right) = \frac{2E^2}{m^2}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right)$$

(T) $\Rightarrow$

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2 m^2}{2|\vec{Q}|^4}\,\frac{2E^2}{m^2}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right)$$

However $\vec{Q}^2 = \vec{p}_f^{\,2} + \vec{p}_i^{\,2} - 2\vec{p}_f \cdot \vec{p}_i = 2p^2 - 2p^2\cos\theta$

$\vec{Q}^2 = 2p^2(1-\cos\theta) = 4p^2\sin^2\theta/2$

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2 \cancel{m^2}}{\cancel{2}\left[4 \cdot p^2\sin^2\frac{\theta}{2}\right]^2}\,\frac{\cancel{2}E^2}{\cancel{m^2}}\left(1 - \beta^2\sin^2\frac{\theta}{2}\right)$$

Coulomb Scattering: The Final Result

XI

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2 E^2}{4p^4 \sin^4\frac{\theta}{2}}\left(1-\beta^2\sin^2\theta/2\right) \Rightarrow$$

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4p^2\beta^2\sin^4\left(\frac{\theta}{2}\right)}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

Coulomb Scattering: Relativistic and non-relativistic approximations

If $\beta \ll 1$ (Non Relativistic limit)

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4p^2v^2}\frac{1}{\sin^4\theta/2} = \frac{Z^2\alpha^2}{4p^4}m^2\frac{1}{\sin^4(\theta/2)}$$

$$p^2 = 2mE \Rightarrow p^4 = 4m^2E^2$$

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2 \cancel{m^2}}{4\cdot 4\cancel{m^2}E^2}\frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \Rightarrow$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha}{4E}\right)^2 \sin^{-4}\left(\frac{\theta}{2}\right) \quad \beta \ll$$

If $\beta \approx 1$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha}{2E}\right)^2\left(\frac{\cos\theta/2}{(\sin\theta/2)^2}\right)^2 \quad \beta \approx 1$$

How to compute cross-sections

RUTHERFORD SCATTERING OF $S=1/2$ PARTICLES FROM A COULOMB POTENTIAL

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4p^2\beta^2\sin^4(\frac{\theta}{2})}\left(1-\beta^2\sin^2\frac{\theta}{2}\right) \Longrightarrow$$

(to get the units correct)

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2\hbar^2}{4p^2\beta^2\sin^4(\theta/2)}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

Assume that the momentum is in GeV/c

PDG $\rightarrow$ $\hbar c = 197.3$ MeV fm

$$\therefore \left(\frac{\hbar c}{\text{GeV}}\right)^2 = \left[\frac{197.3 \text{ MeV fm}}{10^3 \text{MeV}}\right]^2 = \frac{197.3^2\, 10^{-30} m^2}{10^6}$$

$$\left.\begin{array}{l} = 197.3^2\ 10^{-36}\ m^2 \\ 1\ \text{Barn} = 10^{-28} m^2 \end{array}\right\} \Rightarrow$$

$$\left(\frac{\hbar c}{\text{GeV}}\right)^2 = 197.3^2\ 10^{-36}\ 10^{28}\ \text{Barn} = 197.3^2\ 10^{-8}\ B$$

$$= 389.3\ \mu B$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4P^2(\text{GeV})}\ 389.3\mu B\ \frac{1-\beta^2\sin^2\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}$$

- **...and the final result is :**

$$\boxed{\frac{d\sigma}{d\Omega} = 5.2\ \frac{Z^2}{E^2}\ \frac{1-\beta^2\sin\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}\ \mu B}$$

- **These stuff you should know from your undergraduate courses but just in case I review them here........**