Perturbation Theory

Our goal is to evaluate the time-ordered product of fields: (41)

$$\langle 0| T \Phi(x_1)\Phi(y_1)\Phi(x_2)\Phi(y_2)|0\rangle$$

The problem is that we do not know the $\Phi(x_i)$ fields and we would like to express them in terms of the $\Phi_{IN}(x_i)$ fields which we know from the free-field equations. Therefore we define the unitary operator U such that:

$$\left.\begin{aligned}\Phi_{IN}(x) &= U\Phi(x)U^{-1}\\ \Phi(x) &= U^{-1}\Phi_{IN}(x)U\end{aligned}\right\} \quad ①$$

In the Heisenberg picture we have that

$$\dot{\Phi}_{IN} = i\left[H_{IN}(\Phi_{IN},\Pi_{IN}),\Phi_{IN}\right] \quad ②$$

$$\dot{\Phi} = i\left[H(\phi,\pi),\Phi\right] \quad ③$$

$$① \to \dot{\Phi}_{IN} = \dot{U}(t)\Phi U^{-1}(t) + U\dot{\Phi}U^{-1} + U\Phi\dot{U}^{-1} \Rightarrow$$

$$\dot{\Phi}_{IN} = \dot{U}U^{-1}\Phi_{IN}\cancel{UU^{-1}} + U\dot{\Phi}U^{-1} + \cancel{UU^{-1}}\Phi_{IN}U\dot{U}^{-1} \to$$

$$\left.\begin{aligned}&\dot{\Phi}_{IN} = \dot{U}U^{-1}\Phi_{IN} + \Phi_{IN}U\dot{U}^{-1} + U\dot{\Phi}U^{-1}\\ &\text{but}\quad UU^{-1} = 1 \Rightarrow \dot{U}U^{-1} = -U\dot{U}^{-1}\end{aligned}\right\} \Rightarrow$$

(42)

$$\dot{\Phi}_{IN} = \dot{U}U^{-1}\Phi_{IN} - \Phi_{IN}\dot{U}U^{-1} + U\dot{\Phi}U^{-1} \to$$

$$\dot{\Phi}_{IN} = [\dot{U}U^{-1},\Phi_{IN}] + U\dot{\Phi}U^{-1} \xrightarrow{③}$$

$$\dot{\Phi}_{IN} = [\dot{U}U^{-1},\Phi_{IN}] + U\{i[H(\phi,\pi),\Phi]\}U^{-1} \to$$

$$\left.\begin{aligned}&\dot{\Phi}_{IN} = [\dot{U}U^{-1},\Phi_{IN}] + i[H(\phi_{IN},\Pi_{IN}),\Phi_{IN}]\\ &\text{Define}\quad H(\phi_{IN}\,\Pi_{IN}) = H_{IN}(\phi_{IN},\Pi_{IN}) + H_I(\phi_{IN}\Pi_{IN})\\ &\qquad\qquad\qquad\qquad = H_{FREE}(\phi_{IN},\Pi_{IN}) + H_I(\phi_{IN},\Pi_{IN})\end{aligned}\right\} \Rightarrow$$

$$\cancel{\dot{\Phi}_{IN}} = [\dot{U}U^{-1},\Phi_{IN}] + \underbrace{i[H_{IN}(\phi_{IN},\Pi_{IN}),\Phi_{IN}]}_{\cancel{\dot{\Phi}_{IN}}} + i[H_I(\phi_{IN},\Pi_{IN}),\phi_{IN}]$$

$$\therefore\ [\dot{U}U^{-1} + iH_I(\phi_{IN},\Pi_{IN}),\Phi_{IN}] = 0$$

$$\therefore\ -i\dot{U}U^{-1} + H_I(\Phi_{IN},\Pi_{IN}) = 0$$

$$\boxed{i\frac{\partial U}{\partial t} = H_I(\phi_{IN},\Pi_{IN})\cdot U(t)}$$

This equation can be solved as follows: (43)

$$\frac{\partial \mathcal{U}}{\partial t} = -i\, H_I(t)\,\mathcal{U}(t) \Rightarrow$$

$$\left.\int_{t_1}^{t_2} \frac{\partial \mathcal{U}}{\partial t}\,dt = -i \int_{t_1}^{t_2} H_I(t)\,\mathcal{U}(t)\,dt \quad \text{set } \mathcal{U}(t_1)=1 \right\} \Rightarrow$$

$$\mathcal{U}(t_2) = 1 - i\int_{t_1}^{t_2} H_I(t)\,\mathcal{U}(t)\,dt$$

$$\begin{matrix} t_1 \to -\infty \\ t_2 = t \end{matrix} \Rightarrow \mathcal{U}(t) = 1 - i\int_{-\infty}^{t} H_I(t)\,\mathcal{U}(t)\,dt$$

$$\mathcal{U}(t) = 1 - i\int_{-\infty}^{t} H_I(t')\,dt' + (-i)^2 \int_{-\infty}^{t} H_I(t') \int_{-\infty}^{t'} H_I(t'')\,dt'\,dt'' + \dots$$

$$t > t' > t''$$

At the end $\mathcal{U}(t) = T\, e^{-i\int_{-\infty}^{t} H_I(t)\,dt}$ or

$$\mathcal{U}(t) = T\, e^{-i\int d^4x\, \mathcal{H}_I}$$

(where H_I is the Hamiltonian and $\mathcal{H}$ is the Hamiltonian density)

Now we can go back to the time ordered product (44)

$$\langle 0|T\{\Phi(x_1)\Phi(y_1)\Phi(x_2)\Phi(y_2)|0\rangle =$$

$$\langle 0|T\{\mathcal{U}^{-1}(x_1^0)\,\Phi_{IN}\,\mathcal{U}(x_1^0)\,\mathcal{U}^{-1}(y_1^0)\,\Phi_{IN}(y_1)\,\mathcal{U}(y_1^0)$$
$$\mathcal{U}^{-1}(x_2^0)\,\Phi_{IN}(x_2)\,\mathcal{U}(x_2^0)\,\mathcal{U}^{-1}(y_2^0)\,\Phi_{IN}(y_2)\,\mathcal{U}(y_2)$$
$$\}|0\rangle$$

Since this is a time ordered product, inside the product we can combine terms as we want. Assume for example $x_1^0 > y_1^0 > x_2^0 > y_2^0$.

At the end we have that:

$$\langle 0|T(\Phi(x_1)\Phi(y_1)\Phi(x_2)\Phi(y_2)|0\rangle =$$
$$\langle 0|T\{\Phi_{IN}(x_1)\Phi_{IN}(x_2)\Phi_{IN}(y_1)\Phi_{IN}(y_2)\, e^{-i\int d^4x\, \mathcal{H}_I}\}|0\rangle$$

So if we have the Φ_{IN} fields that we get from the free field equations and the interaction part $\mathcal{H}_I$ of the Hamiltonian then we can calculate the amplitude for the process.

Interaction Lagrangian and Disconnected Diagrams

Recall that $\mathcal{L} = \mathcal{L}_{FREE} - \frac{\lambda}{4!}\Phi^4 \Rightarrow$ (45)

$$\mathcal{H}_I(\phi) = -\frac{\lambda}{4!}\Phi^4$$

$$\langle 0|T\left[\Phi_{IN}(x_1)\Phi_{IN}(y_1)\Phi_{IN}(x_2)\Phi_{IN}(y_2)\, e^{-i\int d^4x\, \mathcal{H}_I(x)}\right]|0\rangle =$$

$$= \langle 0|T\Big[\Phi_{IN}(x_1)\Phi_{IN}(y_1)\Phi_{IN}(x_2)\Phi_{IN}(y_2)\Big(1 - i\int d^4x\, \mathcal{H}(x) + \frac{(-i)^2}{2!}\int d^4x \int d^4y\, \mathcal{H}(x)\mathcal{H}(y) \ldots\Big)\Big]|0\rangle$$

The terms of order λ^0 are

$$G^{(0)} = \langle 0|T\left(\phi_{IN}(x_1)\Phi_{IN}(x_2)\Phi_{IN}(y_1)\phi_{IN}(y_2)\right)|0\rangle$$

(WICK'S THEOREM)

$$= \underbrace{\langle 0|N(\phi_{IN}(x_1)\ldots\phi_{IN}(y_2))|0\rangle}_{\to 0} + \overbrace{\langle 0| \Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(x_2)\, N(\Phi_{IN}(y_1)\Phi_{IN}(y_2)) + \ldots\ldots}^{\to 0}$$

many terms with $N(\ )$

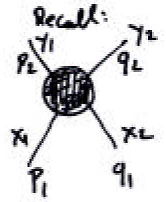
$$\left.\begin{aligned}&\Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(y_1)\Phi^{\circ\circ}_{IN}(x_2)\Phi^{\circ\circ}_{IN}(y_2) +\\ &\Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(x_2)\Phi^{\circ\circ}(y_1)\Phi^{\circ\circ}(y_2) +\\ &\Phi^{\circ}_{IN}(x_1)\Phi^{\circ}_{IN}(y_2)\Phi^{\circ\circ}(x_2)\Phi^{\circ\circ}(y_1)\,|0\rangle\end{aligned}\right\}\ \text{Non vanishing yet}$$

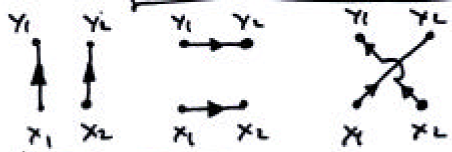
Recall that the contracted operator product for scalar fields was: (46)

$$\varphi^{\circ}(x)\varphi^{\circ}(y) = D_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k\, \frac{i}{k^2-m^2}\, e^{-ik(x-y)}$$

Then: $G^{(0)} = D_F(x_1-y_1)D_F(x_2-y_2) + D_F(x_1-x_2)D_F(y_1-y_2) +$

$D_F(x_1-y_2)\,D_F(x_2-y_1)$





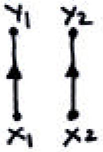
The D_F products can be written as diagrams

so the contribution of these diagrams to the amplitude is:

$$\langle p_2 q_2\, out|p_1 q_1\, in\rangle = \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2 \times$$

$$\times \frac{q_1^2-m^2}{\sqrt{(2\pi)^3 2E_{q_1}}} \times \frac{q_2^2-m^2}{\sqrt{(2\pi)^3 2E_{q_2}}} \times \frac{p_1^2-m^2}{\sqrt{(2\pi)^3 2E_{p_1}}} \times \frac{p_2^2-m^2}{\sqrt{(2\pi)^3 2E_{p_2}}}\, e^{-i(q_1x_2+p_1x_1-p_2y_1-q_2y_2)} \times$$

$$\times \left\{D_F(x_1-y_1)D_F(x_2-y_2) + D_F(x_1-x_2)D_F(y_1-y_2) + D_F(x_1-y_2)D_F(x_2-y_1)\right\}$$

Lets Calculate the  contribution: (47)

$$T_1^{(0)} = \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2 \times \frac{q_1^2-m^2}{\sqrt{2E_{q_1}(2\pi)^3}} \times \frac{q_2^2-m^2}{\sqrt{2E_{q_2}(2\pi)^3}} \times \frac{p_1^2-m^2}{\sqrt{2E_{p_1}(2\pi)^3}} \times \frac{p_2^2-m^2}{\sqrt{2E_{p_2}(2\pi)^3}} \times$$

$$\times\, e^{-i(q_1x_2+p_1x_1-p_2y_1-q_2y_2)} \times \frac{1}{(2\pi)^4}\int d^4k_1 \frac{i}{k_1^2-m^2} e^{-ik_1(x_1-y_1)} \times$$

$$\times \frac{1}{(2\pi)^4}\int d^4k_2 \frac{i}{k_2^2-m^2} e^{-ik_2(x_2-y_2)} \Rightarrow$$

$$T_1^{(0)} = \left(\frac{i}{\sqrt{Z}}\right)^4 \frac{q_1^2-m^2}{\sqrt{2E_{q_1}(2\pi)^3}} \times \frac{q_2^2-m^2}{\sqrt{2E_{q_2}(2\pi)^3}} \times \frac{p_1^2-m^2}{\sqrt{2E_{p_1}(2\pi)^3}} \times \frac{p_2^2-m^2}{\sqrt{2E_{p_2}(2\pi)^3}} \times \left(\frac{1}{(2\pi)^4}\right)^2 \times$$

$$\times \int d^4k_1 \frac{i}{k_1^2-m^2} \int d^4k_2 \frac{i}{k_2^2-m^2} \times \underbrace{\int d^4x_1\, e^{-i(k_1+p_1)\cdot x_1}}_{(2\pi)^4\delta^{(4)}(k_1+p_1)} \times$$

$$\underbrace{\int d^4x_2\, e^{-i(k_2+q_1)x_2}}_{(2\pi)^4\delta^4(k_2+q_1)} \int d^4y_1\, e^{+i(p_2+k_1)y_1} \int d^4y_2\, e^{i(q_2+k_2}$$

$$\Rightarrow T_1^{(0)} = \left(\frac{i}{\sqrt{Z}}\right)^4 \left(\cancel{\frac{1}{(2\pi)^4}}\right)^2 \cancel{\left((2\pi)^4\right)^2}\; \frac{\cancel{q_1^2-m^2}}{\sqrt{}} \times \frac{q_2^2-m^2}{\sqrt{}} \times \frac{\cancel{p_1^2-m^2}}{\sqrt{}} \times \frac{p_2^2-m^2}{\sqrt{}} \times$$

$$\times \frac{i}{\cancel{p_1^2-m^2}} \times \frac{i}{\cancel{q_1^2-m^2}} \underbrace{\int d^4y_1\, e^{i(p_2-p_1)}}_{(2\pi)^4\delta^4(p_2-p_1)} \underbrace{\int d^4y_2\, e^{i(q_2-q_1)}}_{(2\pi)^4\delta(q_1-q_2)}$$

(48)

$$T_1^{(0)} = \left(\frac{i}{\sqrt{Z}}\right)^4 \left((2\pi)^4\right)^2 \delta^{(4)}(p_2-p_1)\,\delta^{(4)}(q_2-q_1) \times$$

$$\times\, i^2 \frac{q_2^2-m^2}{\sqrt{2E_{q_2}(2\pi)^3}} \times \frac{p_2^2-m^2}{\sqrt{2E_{p_2}-m^2}}$$

Since $q_2, p_2 \to m^2$ (real particles at initial + final states)

We see that $T_1^{(0)} = 0$

In fact for the same reason the other two diagrams vanish (not enough q^2-m^2 factors in the denominator). And $G^{(0)} = 0$

Order ? Connected Diagram

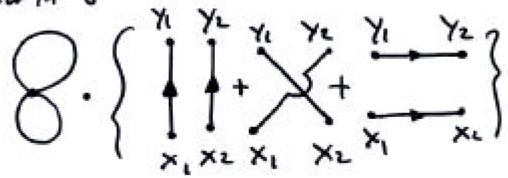
Next calculate the term which is of order (49)
λ^1.

$$G^{(1)} = -\frac{i}{4!}\lambda\int d^4x\,\langle 0|T\,\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Phi^4(x)|0\rangle$$

using Wick's theorem, the N terms vanish and we get:

$$T(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Phi^4(x)) =$$

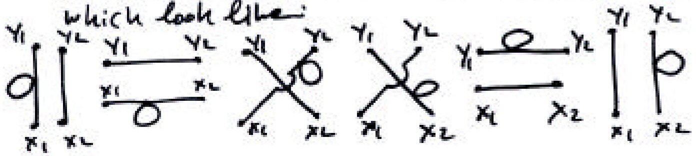
$$\underbrace{3\,\Phi^{\circ}(x)\Phi^{\circ}(x)\Phi^{\circ\circ}(x)\Phi^{\circ\circ}(x)}_{\text{3 ways of doing this as we saw in } G^{(0)}}\cdot\underbrace{G^{(0)}(x_1x_2y_1y_2)}_{\text{as before}} + \text{--- more terms written below}$$

$= 3$ 

$+$

$$+\binom{4}{2}\varphi^{\circ}(x)\varphi^{\circ}(x)\left[2\Phi^{\bullet\bullet}(x)\Phi^{\bullet\bullet}(x_1)\Phi^{\circ\circ\circ}(x)\Phi^{\circ\circ\circ}(x_2)\right]\Phi^{\bullet\bullet\bullet\bullet}(y_1)\Phi^{\bullet\bullet\bullet\bullet}(y_2)$$

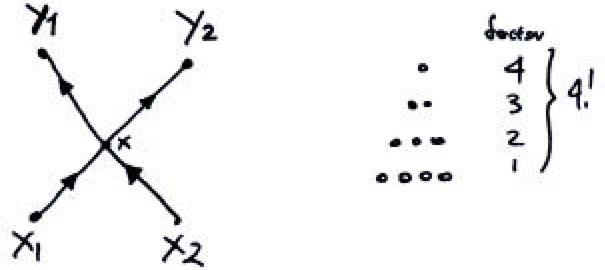
$+ \ldots$ a class of terms of the type $\phi(x)\phi(x)[\quad]\phi(y_i)\phi(y_j)$ which look like:



The previous two classes of diagramms (50)
contain disconnected diagramms and as we already know vanish (not enough denominators). But there is also one more order λ graph which is:

$$4!\;\phi^{\circ}(x)\phi^{\circ}(x_1)\phi^{\circ\circ}(x)\phi^{\circ\circ}(y_1)\phi^{\circ\circ\circ}(x)\phi^{\circ\circ\circ}(y_2)\Phi^{\circ\circ\circ\circ}(x)\phi^{\circ\circ\circ\circ}(x_2)$$

which can be written as



and does have enough $\frac{1}{k^2-m^2}$ terms ...

This term can be evaluated as follows.

Result and Higher Order Terms

(51)

$$G^{(4)}=\left(\frac{i}{\sqrt{2}}\right)^4\iint d^4y_1\int d^4y_2\int d^4x_1\int d^4x_2\; \underbrace{e^{-i(q_1x_2+p_1x_1-q_2y_2-p_2y_1)}}_{\text{this is from } \int d^4y_1}\, d^4x \;\times$$

$$\times\frac{p_1^2-m^2}{\sqrt{2E_{p_1}(2\pi)^3}}\times\frac{p_2^2-m^2}{\sqrt{2E_{p_2}(2\pi)^3}}\times\frac{q_1^2-m^2}{\sqrt{2E_{q_1}(2\pi)^3}}\times\frac{q_2^2-m^2}{\sqrt{2E_{q_2}(2\pi)^3}}\times\int d^4k_1\int d^4k_2\int d^4k_3\int d^4k_4$$

$$4!\times\left(\frac{i}{(2\pi)^4}\right)^4\underbrace{\frac{e^{ik_1(x_1-x)}}{k_1^2-m^2}\times\frac{e^{ik_2(x_2-x)}}{k_2^2-m^2}\times\frac{e^{ik_3(y_1-x)}}{k_3^2-m^2}\times\frac{e^{ik_4(y_2-x)}}{k_4^2-m^2}}_{\text{PROPAGOTORS OF S=0 FIELDS}}\times$$

$$\times\frac{-i}{4!}\times\lambda$$

(1) The $e^{ix_i\cdot k_i}$ terms will pair with the $e^{iq_ix_i}$ terms to produce $\delta^4(\,)\times(2\pi)^4$ terms (4 of them) and each time they do it they will "spend" a $\int d^4x_i$.

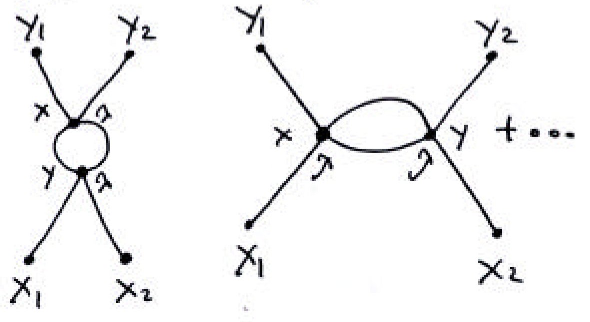
(2) the $\delta^4()$ functions then will kill one $\int d^4k_i$ term leaving at the end $\int d^4x\, e^{-i(k_1+k_2-k_3-k_4)x}$

At the end we have:

$$\boxed{\langle p_2q_2\,\text{out}|p_1q_1\,\text{in}\rangle=\frac{-i\lambda}{2!}\,\frac{(2\pi)^4\,\delta(q_1+p_1-q_2-p_2)}{\sqrt{2E_{q_1}(2\pi)^3}\sqrt{2E_{q_2}(2\pi)^3}\sqrt{2E_{p_1}(2\pi)^3}\sqrt{2E_{p_2}(2\pi)^3}}}$$

(52)

At λ^2-order things will look like



which may be written as

$$\frac{(-i)^2}{2!}\left(\frac{\lambda}{4!}\right)\int d^4x\int d^4y\;\times$$

$$\langle 0|T(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Phi^4(x)\Phi^4(y))|0\rangle$$

Clearly we need some rules to calculate these diagrams so we dont have to repeat all these calculations every time we want to evaluate the cross section of a process.