

Wick's Theorem



Normal Ordering and Contracted Product of Operators. WICKS THEOREM: (34)

IN ORDER TO CALCULATE $\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$ we need to know a few more "tricks"....

Recall that $\phi(x) = \int d^3k [a_k f_k(x) + a_k^\dagger f_k^*(x)]$ with

$$f_k(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ikx} \quad (\text{positive freq. part})$$

③ DEFINE THE N OPERATOR WHICH ORDERS ANNIHILATION OPERATORS (positive frequency parts) TO THE RIGHT OF THE CREATION OPERATORS. THIS IS CALLED "NORMAL ORDERING" OR "NORMAL PRODUCT". It works like this:

$$\begin{aligned} N(\phi(x)\phi(y)) &= N((\overset{\text{positive freq.}}{a^\dagger(x)} + a(x))(\overset{\text{neg. freq.}}{a^\dagger(y)} + a(y))) \\ &= N(\overset{\vee}{a^\dagger(x)} \overset{\vee}{a^\dagger(y)} + \overset{\vee}{a^\dagger(x)} \overset{\vee}{a(y)} + \overset{\vee}{a(x)} \overset{\vee}{a^\dagger(y)} + \overset{\vee}{a(x)} \overset{\vee}{a(y)}) \\ &= \overset{\vee}{a^\dagger(x)} \overset{\vee}{a^\dagger(y)} + \overset{\vee}{a^\dagger(x)} \overset{\vee}{a(y)} + \overset{\vee}{a(x)} \overset{\vee}{a^\dagger(y)} + \overset{\vee}{a(x)} \overset{\vee}{a(y)} \end{aligned}$$

Some times $N(\phi(x)\phi(y))$ is written as

$:\phi(x)\phi(y):$ and it is one and the same thing.

Define the contracted field product by: (35)

$$\phi(x)\phi(y) \equiv T(\phi(x)\phi(y)) - N(\phi(x)\phi(y))$$

Lets try to calculate $\phi(x)\phi(y)$ for $x^0 > y^0$

$$\begin{aligned} \phi(x)\phi(y) &= T\{(\Phi^{(+)}(x) + \Phi^{(-)}(x)) \cdot (\Phi^{(+)}(y) + \Phi^{(-)}(y))\} + \\ &\quad - N\{(\Phi^{(+)}(x) + \Phi^{(-)}(x)) \cdot (\Phi^{(+)}(y) + \Phi^{(-)}(y))\} \Rightarrow \end{aligned}$$

$$\begin{aligned} \phi(x)\phi(y) &= \cancel{\Phi^{(+)}(x)\Phi^{(+)}(y)} + \cancel{\Phi^{(+)}(x)\Phi^{(-)}(y)} + \overset{y^0 < x^0}{\text{CONTIN. TO T.C.}} \Phi^{(-)}(x)\Phi^{(+)}(y) + \cancel{\Phi^{(-)}(x)\Phi^{(+)}(y)} + 0 + \\ &\quad - (\cancel{\Phi^{(+)}(x)\Phi^{(+)}(y)} + \cancel{\Phi^{(+)}(x)\Phi^{(-)}(y)} + \cancel{\Phi^{(-)}(x)\Phi^{(+)}(y)} + \cancel{\Phi^{(-)}(x)\Phi^{(-)}(y)}) \Rightarrow \end{aligned}$$

$$\phi(x)\phi(y) = \Phi^{(+)}(x)\Phi^{(-)}(y) - \Phi^{(-)}(y)\Phi^{(+)}(x) \Rightarrow$$

$$\boxed{\phi(x)\phi(y) = [\Phi^{(+)}(x), \Phi^{(-)}(y)]}$$



So

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$$\Phi(x)\Phi(y) = [\Phi(x), \Phi(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2E_k}} \frac{d^3k'}{\sqrt{2E_{k'}}} e^{-ikx} e^{ik'y} [\alpha_k, \alpha_{k'}]$$

Since $\Phi(x)$ is a quantum field we have that

$$[\alpha_{\vec{k}}, \alpha_{\vec{k}'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}')$$

Therefore:

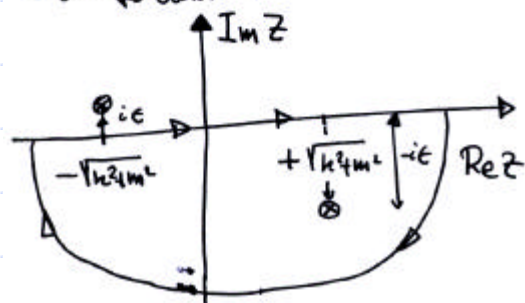
$$\Phi(x)\Phi(y) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2E_k}} \frac{d^3k'}{\sqrt{2E_{k'}}} \delta^3(\vec{k} - \vec{k}') e^{-ikx + ik'y}$$

$$\text{and } \Phi(x)\Phi(y) = [\Phi(x), \Phi(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2E_k}} e^{-ik \cdot (x-y)} \quad (A)$$

$$\text{But } D_F(x) = \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 - m^2} e^{-ikx} = \frac{i}{(2\pi)^4} \int d^3k \int dk^0 \frac{e^{-ikx}}{k^2 - m^2} \Rightarrow$$

$$D_F(x) = \frac{i}{(2\pi)^4} \int d^3k e^{i\vec{k} \cdot \vec{x}} \underbrace{\int_{-\infty}^{\infty} dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - \sqrt{k^2 + m^2})(k^0 + \sqrt{k^2 + m^2})}}_I \quad (C)$$

Recall from Complex analysis that I can be evaluated as follows



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$$I = \int_{-\infty}^{\infty} dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - \sqrt{k^2 + m^2} + i\epsilon)(k^0 + \sqrt{k^2 + m^2} - i\epsilon)}$$

$$I = -2\pi i \text{Res} \left\{ \frac{e^{-ik^0 x^0}}{(k^0 - \sqrt{k^2 + m^2} + i\epsilon)(k^0 + \sqrt{k^2 + m^2} - i\epsilon)} \right\}$$

$$I = -2\pi i \left\{ \frac{e^{-ik^0 x^0}}{(k^0 - \sqrt{k^2 + m^2} + i\epsilon)(k^0 + \sqrt{k^2 + m^2} - i\epsilon)} \right\}_{k^0 = \omega_k}$$

$$I = -2\pi i \frac{e^{-i\omega_k x^0}}{2\omega_k} \quad (B)$$

$$(B) + (C) \Rightarrow D_F(x) = \frac{i}{(2\pi)^4} \int d^3k (-2\pi i) e^{i\vec{k} \cdot \vec{x}} \frac{e^{-i\omega_k x^0}}{2\omega_k} \Rightarrow$$

$$D_F(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} e^{-ikx} \text{ or } D_F = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2E_k} e^{-ikx}$$

In conclusion:

$$\Phi(x)\Phi(y) = [\Phi(x), \Phi(y)] = \frac{1}{(2\pi)^4} \int d^4k e^{-ikx} \frac{i}{k^2 - m^2}$$

$\frac{i}{k^2 - m^2}$ is the scalar field propagator!!



Summary

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Wick's Theorem

G.C. Wick, Phys. Rev. 80,
P268, 1950

If one wonders by now why are all these
useful here is the answer:

Our goal was to evaluate the Vacuum
expectation value of the time ordered
product: $\langle 0 | T \phi(x_1) \phi(x_2) \phi(y_1) \phi(y_2) | 0 \rangle$

The contracted operator formula gives us
almost what we want:

$$\langle 0 | T \phi(x_1) \phi(y_1) | 0 \rangle = \underbrace{\langle 0 | \phi^{\circ}(x_1) \phi^{\circ}(y_1) | 0 \rangle}_{\text{we know by now what is this and how is it calculated}} +$$

$$+ \underbrace{\langle 0 | N(\phi(x_1) \phi(y_1)) | 0 \rangle}_{\rightarrow 0 \text{ since } a_n | 0 \rangle = 0}$$

We only need to make this more general
so it applies to 4-fields. To do this
we use the theorem discovered by
G.C. Wick

$$\begin{aligned} T(\phi(x_1) \phi(x_2) \dots \phi(x_n)) = & N(\phi(x_1) \dots \phi(x_n)) + \\ & + \sum_{i < j} \phi^{\circ}(x_i) \phi^{\circ}(x_j) N(\phi(x_1) \dots \phi(x_n))_{\substack{a \dots b \neq i, j \\ a \dots b \neq k, l}} + \\ & + \sum_{\substack{i < j \\ k < l}} \phi^{\circ}(x_i) \phi^{\circ}(x_j) \phi^{\circ}(x_k) \phi^{\circ}(x_l) N(\phi(x_1) \dots \phi(x_n))_{\substack{a \dots b \neq i, j \\ a \dots b \neq k, l}} + \\ & + \dots \end{aligned}$$

Examples



Examples:

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$$1. T(\phi) = \phi$$

$$2. T(\phi(x)\phi(y)) = \phi^{\circ}(x)\phi^{\circ}(y) + N(\phi(x)\phi(y))$$

$$3. T(\phi(x)\phi(y)\phi(z)) = N(\phi(x)\phi(y)\phi(z)) + \\ + \phi^{\circ}(x)\phi^{\circ}(y)\phi^{\circ}(z) + \\ + \phi^{\circ}(x)\phi^{\circ}(z)\phi^{\circ}(y) + \\ + \phi^{\circ}(y)\phi^{\circ}(z)\phi^{\circ}(x)$$

$$4. T(\phi_1\phi_2\phi_3\phi_4) = N(\phi_1\phi_2\phi_3\phi_4) + \\ \phi_i = \phi(x_i) \quad + \phi_1^{\circ}\phi_2^{\circ}N(\phi_3\phi_4) + \phi_1^{\circ}\phi_3^{\circ}N(\phi_2\phi_4) + \\ + \phi_1^{\circ}\phi_4^{\circ}N(\phi_2\phi_3) + \phi_2^{\circ}\phi_3^{\circ}N(\phi_1\phi_4) + \\ + \phi_2^{\circ}\phi_4^{\circ}N(\phi_1\phi_3) + \phi_3^{\circ}\phi_4^{\circ}N(\phi_1\phi_2) \\ + \phi_1^{\circ}\phi_2^{\circ}\phi_3^{\circ}\phi_4^{\circ} + \phi_1^{\circ}\phi_3^{\circ}\phi_2^{\circ}\phi_4^{\circ} + \\ + \phi_1^{\circ}\phi_4^{\circ}\phi_2^{\circ}\phi_3^{\circ} +$$