From Free to Interacting Fields: S-Matrix

So far we considered Lagrangians that (25)
(E) did not allow for field-field interactions.
But in reality the fields interact both with each other but also with themselves.

Consider for example:

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2}_{\text{Free-}\mathcal{L}} + \underbrace{\frac{\lambda}{4}\phi^4}_{\mathcal{L}_{\text{INTERACTION}}}$$

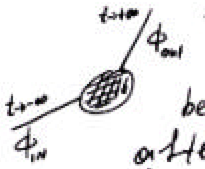
using the Lagrange equations you get that

$$(\Box + m^2)\,\varphi(x) = J(x) \quad (1)$$

$$J(x) = \lambda\phi^3 \quad (2)$$

obviously this is a complicated Diff. Equation to solve and we have to find a way out of this problem.

S-Matrix



Lets assume that the fields before they interact ($t\to-\infty$) and after they interact ($t\to+\infty$) are free ϕ_{in}, ϕ_{out} fields respectively

So for $t\to-\infty$ (26)

$$(\Box + m^2)\,\phi_{in}(x) = 0$$

$$\varphi_{in}(x) = \int d^3k\left[a_{in}(\vec{k})\,f_{\vec{k}}(x) + a^{\dagger}_{in}(\vec{k})\,f^{*}_{\vec{k}}(x)\right]$$

$$f_{\vec{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}}\,e^{-ikx} \quad \text{(positive freq.)}$$

and for $t\to+\infty$

$$(\Box + m^2)\,\phi_{out}(x) = 0$$

$$\varphi_{out}(x) = \int d^3k\left[a_{out}(\vec{k})\,f_{\vec{k}}(x) + a^{\dagger}_{out}\,f^{*}_{k}(x)\right]$$

As in Quantum Mechanics we always want to calculate:

$$S_{\alpha\beta} = \langle \beta\,\mathrm{out} | \alpha_{in} \rangle$$

$$|\alpha\ \mathrm{in}\rangle = |p_1 p_2 \ldots p_n\ \mathrm{in}\rangle$$

$$|\beta\ \mathrm{out}\rangle = |p_1' p_2' \ldots p_m'\ \mathrm{out}\rangle$$

and this become $\sigma \sim |\langle\beta|\alpha\rangle_{in}^{out}|^2$

(27)

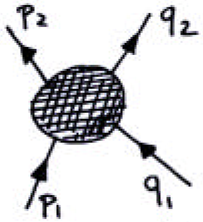
It is more formal to write:

$$\lim_{t\to+\infty} \langle \alpha|\phi(x)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\phi_{out}|\beta\rangle$$

$$\lim_{t\to-\infty} \langle \alpha|\phi(x)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\phi_{IN}|\beta\rangle$$

Consider the interaction:

Suppose $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \underbrace{\frac{\lambda}{4}\phi^4}_{\mathcal{L}_I}$



$$S = \langle p_2 q_2\ out|\, p_1 q_1\ IN\rangle = \langle p_2 q_2\ out|\, a^\dagger_{IN}(p_1)\,|q_1\ IN\rangle$$

$$S = \underbrace{\cancel{\langle p_2 q_2\ out|\, a^\dagger_{out}(p_1)\,|q_1\ IN\rangle}}_{\text{Forward term}} + \langle p_2 q_2\ out|\, a^\dagger_{IN}(p_1) - a^\dagger_{out}(p_1)\,|q_1\ IN\rangle$$

$$\boxed{S = \langle p_2 q_2\ out|\, a^\dagger_{IN}(p_1) - a^\dagger_{out}(p_1)\,|q_1\ IN\rangle} \quad (3)$$

Now we need a way to calculate $a^\dagger_{IN,out}(k)$ as a function of $\Phi(x)$ ...

(28)

Recall:

$$\Phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[a_k e^{-ikx} + a_k^\dagger e^{ikx}\right]$$

But $\int d^3x \frac{e^{ikx}}{\sqrt{(2\pi)^3 2\omega_k}}\, \phi(x) =$

$$\int d^3x \frac{e^{ikx}}{\sqrt{(2\pi)^3 2\omega_k}} \int \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}}\left[a_{k'} e^{-ik'x} + a^\dagger_{k'} e^{ik'x}\right]$$

$$= \int \frac{d^3k'}{\sqrt{2\omega_k 2\omega_{k'}}} \frac{1}{(2\pi)^3}\int d^3x \left[a_{k'}\, e^{i(\vec{k}'-\vec{k})}\, e^{i(\omega_k-\omega_{k'})t} + a^\dagger_{k'}\, e^{-i(\vec{k}'+\vec{k})}\, e^{i(\omega_k+\omega_{k'})t}\right]$$

$$= \int \frac{d^3k'}{\sqrt{2\omega_k 2\omega_{k'}}}\left[a_{k'}\,\delta^3(\vec{k}'-\vec{k})\, e^{i(\omega_k-\omega_{k'})t} + a^\dagger_{k'}\,\delta^3(\vec{k}+\vec{k}')\, e^{i(\omega_k+\omega_{k'})t}\right]$$

$$= \frac{1}{2\omega_k}\left[a(\vec{k}) + a^\dagger(-\vec{k})\, e^{2i\omega_k t}\right]$$

In Conclusion: (29)

$$\int d^3x\, f^*_{\vec{k}}(x)\, \Phi(x) = \frac{1}{2\omega_k}\left[a_{(\vec{k})} + a^\dagger_{(-\vec{k})} e^{2i\omega t}\right]$$

and (trust me)

$$\int d^3x\, f^*_{\vec{k}}(x)\, \partial_0 \phi(x) = -\frac{i}{2}\left[a_{\vec{k}} - a^\dagger_{-\vec{k}} e^{2i\omega t}\right]$$

By solving this you get

$$a(\vec{k}) = \int d^3x \left[f^*_{\vec{k}}(x)\,\omega_k \phi(x) + i f^*_{\vec{k}}(x)\partial_0 \phi(x)\right]$$

recall that $f^*_{\vec{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega}} e^{ikx} \to \partial_0 f^* = i\omega_k f^*$

so

$$a(\vec{k}) = \int d^3x \left[\partial_0 f^*_{\vec{k}} \left(\tfrac{1}{i}\right)\phi + i\,\partial_0\phi\, f^*_{\vec{k}}(x)\right]$$

$$\boxed{\begin{aligned} a(\vec{k}) &= i\int d^3x \left(f^*\partial_0\phi - \phi\partial_0 f^*\right) \\ a(\vec{k}) &= i\int d^3x\, f^*_{\vec{k}}(x)\overleftrightarrow{\partial_0}\phi(x) \end{aligned}}$$

$$f^*_{\vec{k}}(x)\overleftrightarrow{\partial_0}\phi(x) = f^*_{\vec{k}}(x)\,\partial_0\phi(x) - \partial_0 f^*_{\vec{k}}\,\phi(x)$$

Now we can go back to calculate S (30)

$$S = \langle p_2 q_2\, out |\, a^\dagger_{in}(p_1) - a^\dagger_{out}(p_1)\, | q_1\, in\rangle$$

$$S = -i\langle p_2 q_2 | \int d^3x\, f_{\vec{p}_1}\overleftrightarrow{\partial_0}\left[\phi_{in}(x) - \phi_{out}(x)\right] | q_1\, in\rangle$$

$$S = \frac{-i}{Z^{1/2}}\langle p_2 q_2 | \left(\lim_{x^0\to-\infty} - \lim_{x^0\to+\infty}\right)\int d^3x\, f_{\vec{p}_1}\overleftrightarrow{\partial_0}\phi(x) | q_1\, in\rangle$$

But $\left(\lim_{x^0\to+\infty} - \lim_{x^0\to-\infty}\right)\int d^3x\, g(x) = \int d^4x \frac{\partial g}{\partial x^0}$

$$\text{so } S = \frac{i}{Z^{1/2}}\int d^4x \frac{\partial}{\partial x^0}\left\{f_{\vec{p}_1}(x)\overleftrightarrow{\partial_0}\phi(x)\right\} | q_1\, in\rangle$$

$$S = \frac{i}{Z^{1/2}}\int d^4x \langle p_2 q_2\, out | f_{\vec{p}_1}\frac{\partial^2\phi(x)}{\partial x_0^2} - \frac{\partial^2 f_{\vec{p}_1}(x)}{\partial x_0^2}\phi(x) | q_1\, in\rangle$$

integrate by parts

$$S = \frac{i}{Z^{1/2}}\int d^4x \langle p_2 q_2\, out | f_{\vec{p}_1}\frac{\partial^2\phi(x)}{\partial x_0^2} - \phi(\vec{\nabla}^2 - m^2) f_{\vec{p}_1}(x) | q_1\, in\rangle$$

$$S = \frac{i}{\sqrt{Z}}\int d^4x_1 \langle p_2 q_2\, out | \phi(x_1) | q_1\, in\rangle \overleftarrow{(\Box_{x_1} + m^2)} f_{\vec{p}_1}(x_1)$$

$$S = \frac{i}{\sqrt{Z}}\int d^4x_1\, f_{\vec{p}_1}(x_1)\, \overrightarrow{(\Box_{x_1} + m^2)} \langle p_2 q_2\, out | \phi(x_1) | q_1\, in\rangle$$

The Time Ordered Product

31

and

$$\langle p_2 q_2\, out | p_1 q_1\, in\rangle = \frac{i}{\sqrt{Z}}\int d^4x\, f_p(x)(\Box_x+m^2)\langle p_2 q_2\, out|\phi(x)|q_1\, in\rangle \quad (A)$$

Next we will remove the p_2 out state from

$$\langle p_2 q_2\, out|\phi(x)|q_1\, in\rangle = \langle q_2\, out|\, a_{out}(p_2)\,\phi(x)|q_1\, in\rangle$$

add and sub. $\langle|\Phi a_{in}(p_2)|\rangle$

$$= \cancel{\langle q_2\, out|\,\Phi\, a_{in}(p_2)|q_1\, in\rangle} + \langle q_2\, out|\, a_{out}(p_2)\Phi(x) - \Phi(x)\, a_{in}(p_2)|q_1\, in\rangle$$

there is no p_2 state with input to be killed

$$= \langle q_2\, out|\, a_{out}(p_2)\Phi(x) - \Phi(x)\, a_{in}(p_2)|q_1\, in\rangle$$

$$= +i\int d^3y\, \langle out\, q_2|\,\phi_{out}(y)\Phi(x) - \Phi(x)\Phi_{in}(y)|q_1\, in\rangle \times \overleftrightarrow{\partial}_{y_0} f^*_{\vec{p}_2}(y) \quad (B)$$

Define the <u>time ordered product</u> of two fields as:

$$T(a(x)b(y)) = a(x)\, b(y)\,\Theta(t_x - t_y) + b(y)\, a(x)\,\Theta(t_y - t_x)$$

with

$$\Theta(t_i - t_j) = \begin{cases} 1 & t_i > t_j \\ 0 & t_i < t_j \end{cases}$$

32

Now we can take again as before the limits for

$t \to -\infty$ which will replace Φ_{in} with Φ and
$t \to +\infty$ which will replace Φ_{out} with Φ.

$$(B) \to \langle p_2 q_2\, out|\Phi(x)|q_1\, in\rangle =$$

$$+\frac{i}{\sqrt{Z}}\left(\lim_{y^0\to+\infty} - \lim_{y^0\to-\infty}\right)\int d^3y\, \langle out\, q_2|T\Phi(y)\Phi(x)|q_1\, in\rangle \times \overleftrightarrow{\partial}_{y_0} f^*_{\vec{p}_2}(x)$$

as you can see for $y^0 \to +\infty$ $T(\Phi(y)\Phi(x)) = \Phi(y)\Phi(x)$ and
for $y^0 \to -\infty$ $T(\Phi(y)\Phi(x)) = \Phi(x)\Phi(y)$,

As before $\langle p_2 q_2\, out|\Phi(x)|q_1\, in\rangle =$

$$\frac{i}{\sqrt{Z}}\int d^4x\, \frac{\partial}{\partial x^0}\left\{\langle out\, q_2|\, T(\Phi(y)\Phi(x))\,|q_1\, in\rangle \times \overleftrightarrow{\partial}_{y_0} f^*_{\vec{p}_2}(x)\right\} \quad (C)$$

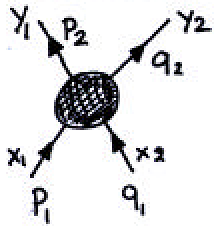
Finally using (A) (C) we get that

$$\langle p q_2\, out|p_1 q_1\, in\rangle = \left(\frac{i}{\sqrt{Z}}\right)^2\int d^4y\int d^4x\, f_{\vec{p}_1}(x)\, f^*_{\vec{2}}(y)\, (\Box_x+m^2)(\Box_y+m^2)\langle q_2\, out|T\Phi(y)\Phi(x)|q_1\, in\rangle$$

Summary

SUMMARY: We want to calculate the amplitude (33)

(S-matrix) for the reaction $p_1+q_1 \to p_2+q_2$



where $p_1 p_2 q_1 q_2$ are spin zero (scalar) particles described by a theory with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

We have found that: $\langle p_2 q_2\, \mathrm{out} | p_1 q_1\, \mathrm{in}\rangle =$

$$\left(\frac{i}{\sqrt{z}}\right)^4 \int d^4y_1 \int d^4y_2 \int d^4x_1 \int d^4x_2\, f_{\vec{p}_1}(x_1)\, f_{q_1}(x_2)\, f^*_{p_2}(y_1)\, f^*_{q_2}(y_2)$$

$$\times \left(\Box_{x_1}+m^2\right)\left(\Box_{x_2}+m^2\right)\left(\Box_{y_1}+m^2\right)\left(\Box_{y_2}+m^2\right).$$

$$\langle 0 | T\, \phi(y_1)\,\phi(y_2)\,\phi(x_1)\,\phi(x_2) | 0\rangle$$

By making the $\Box$ operators act on $f(x)$'s (by integrating twice by parts and dropping the surface terms) we get: $\langle p_2 q_2\, \mathrm{out} | p_1 q_1\, \mathrm{in}\rangle =$

$$= \left(\frac{i}{\sqrt{z}}\right)^4 \int d^4y_1 \int d^4y_2 \int d^4x_1 \int d^4x_2\; e^{-i(q_1x_2+p_1x_1-q_2y_2-p_2y_1)}$$

$$\times \frac{(q_1^2-m^2)}{\sqrt{(2\pi)^3 2E_{q_1}}} \times \frac{(q_2^2-m^2)}{\sqrt{(2\pi)^3 2E_{q_2}}} \times \frac{(p_1^2-m^2)}{\sqrt{(2\pi)^3 2E_{p_1}}} \times \frac{(p_2^2-m^2)}{\sqrt{(2\pi)^3 2E_{p_2}}}$$

$$\langle 0 | T\left(\phi(x_1)\,\phi(x_2)\,\phi(y_1)\,\phi(y_2)\right) | 0\rangle$$