Expanding the scalar field in terms of creation and annihilation operators

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The next step now is to quantize the classical fields we studied so far.

Back to the scalar field $\Phi(x)$

Let: $\Phi(x) = \int \frac{d^4k}{(2\pi)^{3/2}} e^{-ik\cdot x} \tilde{\varphi}(k)$ (F.T.)

where $k\cdot x = k^0x^0 - \vec{k}\cdot\vec{x}$

$$(\Box + m^2)\Phi(x) = 0 \Rightarrow$$

$$\int \frac{d^4k}{(2\pi)^{3/2}} \underbrace{(\partial_\mu\partial^\mu + m^2) e^{-ik\cdot x}}_{\{(-ik)\cdot(-ik) + m^2\} e^{-ik\cdot x}} \tilde{\varphi}(k) = 0$$

$$\therefore (k^2 - m^2)\tilde{\varphi}(k) = 0$$

Try $\tilde{\varphi}(k) = \delta(k^2 - m^2)\,\varphi(k)$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4k\, e^{-ik\cdot x} \delta(k^2 - m^2)\varphi(k)$$

remember $\delta(\vec{k} - \vec{k}') = \frac{1}{(2\pi)^3} \int e^{-i\vec{x}(\vec{k} - \vec{k}')} d^3x$

$$\delta(x^2 - a^2) = \frac{1}{2a}\{\delta(x-a) + \delta(x+a)\}$$

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$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{-i(k^0x^0 - \vec{k}\vec{x})} \delta(k^{0\,2} - \vec{k}^2 - m^2)\varphi(k)\, dk^0 d^3\vec{k}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{-i(k^0x^0 - \vec{k}\vec{x})} \varphi(k) \frac{1}{2\sqrt{\vec{k}^2+m^2}} \left\{\delta(k^0 - \sqrt{\vec{k}^2+m^2}) + \delta(k^0 + \sqrt{\vec{k}^2+m^2})\right\} dk^0 d^3k$$

Call $E_k = \sqrt{\vec{k}^2 + m^2}$ and

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \int dk^0\, \varphi(k) \frac{1}{2E_k} \left\{\delta(k^0 - E_k) + \delta(k^0 + E_k)\right\} e^{-ik\cdot x}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} \left\{ \underbrace{\varphi(E_k, \vec{k})}_{\varphi_+} e^{-iE_kx^0 + i\vec{k}\vec{x}} + \underbrace{\varphi(-E_k, \vec{k})}_{\varphi_-} e^{iE_kx^0 + i\vec{k}\cdot\vec{x}} \right\}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} \left[\varphi_+(\vec{k}) e^{-iE_kx^0 + i\vec{k}\cdot\vec{x}} + \varphi_-(\vec{k}) e^{iE_kx^0 + i\vec{k}\cdot\vec{x}} \right] \quad (1)$$

Since we have already in mind that $\Phi(x)$ will become an operator in Quantum Field Theory, we demand that it is Hermitian that is $\Phi^\dagger(x) = \Phi(x) \Rightarrow$

$$\varphi_+^\dagger(\vec{k}) e^{iE_kx^0 - i\vec{k}\vec{x}} + \varphi_-^\dagger(\vec{k}) e^{-iE_kx^0 - i\vec{k}\cdot\vec{x}} =$$

$$\varphi_+(\vec{k}) e^{-iE_kx^0 + i\vec{k}\cdot\vec{x}} + \varphi_-(\vec{k}) e^{iE_kx^0 + i\vec{k}\cdot\vec{x}}$$

$$\varphi_+^\dagger(\vec{k}) = \varphi_-(-\vec{k}) \Rightarrow \boxed{\varphi_-(\vec{k}) = \varphi_+^\dagger(-\vec{k})} \quad (2)$$

The Quantization of the Scalar Field I

①② →

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$$\Phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2E_k}\Big[\varphi_+(\vec{k})e^{-iE_kx^0+i\vec{k}\vec{x}}+\underbrace{\varphi_+^{+}(\vec{k})e^{iE_kx^0+i\vec{k}\vec{x}}}_{\varphi_+^{+}(\vec{k})e^{iE_kx^0-i\vec{k}\vec{x}}}\Big]$$

define $$a_{\vec{k}}=\frac{\varphi_+(\vec{k})}{\sqrt{2E_k}}$$

$$\therefore\ \Phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{(2E_k)^{1/2}}\left\{a_{\vec{k}}e^{-iE_kx^0+i\vec{k}\vec{x}}+a^{+}_{\vec{k}}e^{iE_kx^0-i\vec{k}\vec{x}}\right\}$$

Scalar Field Quantization

1._ Identify Canonical Variables and replace them with operators. In Quantum Field Theory all fields are Operators.

2._ Impose Canonical Commutation relations (Similar to $[\hat{p},\hat{x}]=-i\hbar$ in Quantum Mechanics)

3._ The demand that the field obeys position-momentum-type of Commutation relations results to a Quantized Field (Quantum Field theory)

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At this point recall the theory of the harmonic oscillator in Quantum Mechanics since the procedure to quantize a field is very similar.

Start with $L=\frac{\hat{p}^2}{2m}-\frac{1}{2}m\omega^2\hat{q}^2\Rightarrow H=p\dot{q}-L\Rightarrow$

$$\boxed{\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{q}^2}\quad ①$$

Define $\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q}+\frac{i\hat{p}}{m\omega}\right)$ and $\hat{a}^{+}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q}-\frac{i\hat{p}}{m\omega}\right)$

Demand that $[\hat{q},\hat{p}]=i\hbar$ ②

①② $\Rightarrow$ $H=\hbar\omega\left(\hat{a}^{+}\hat{a}+\frac{1}{2}\right)$ ③

$[\hat{q},\hat{p}]=i\hbar\Leftrightarrow[a,a^{+}]=1$ ④

So by defining a Lagrangian and from it a Hamiltonian we have defined a Classical theory. The requirement $[\hat{q},\hat{p}]=i\hbar$ makes it a Quantum theory because:

Say $H|S\rangle=E_s|S\rangle$, what is $a|S\rangle$? is it also eigenstate of $\hat{H}$

$$H\,a|S\rangle=\hbar\omega\left(a^{+}a+\tfrac{1}{2}\right)a|S\rangle=\frac{\hbar\omega}{2}a|s\rangle+\hbar\omega a^{+}aa|s\rangle$$
$$H\,a|s\rangle=\frac{\hbar\omega}{2}a|s\rangle+\hbar\omega(aa^{+}-1)a|s\rangle$$
$$H\,a|s\rangle=\frac{\hbar\omega}{2}a|s\rangle+\hbar\omega a\,a^{+}a|s\rangle-\hbar\omega\,a|s\rangle$$
$$H\,a|s\rangle=a\hat{H}|s\rangle-\hbar\omega\,a|s\rangle=(E_s-\hbar\omega)\,a|s\rangle\ !$$

The Quantization of the Scalar Field II

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So $\hat{a}|s\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue $E_s - \hbar\omega$

$\hat{a} \rightarrow$ "lowering operator" (annihilation)

$\hat{a}^{\dagger} \rightarrow$ "raising operator" (creation)

Back to the scalar field $\Phi(x)$

Define $\pi(\vec{x},t) = \partial^0 \phi(\vec{x},t)$ (momentum?)

and require $[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$ to quantize the field.

$$\phi(\vec{x}',t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k'}{\sqrt{2E_{k'}}} \left\{ e^{-iE_k'x^0 + i\vec{k}'\cdot\vec{x}'} a_{\vec{k}'} + a^{\dagger}_{k'} e^{iE_k'x^0 - i\vec{k}'\cdot\vec{x}'} \right\}$$

$$\dot{\phi} = \pi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} \left\{ (-i) E_k e^{-iE_k x^0 + i\vec{k}\cdot\vec{x}} a_k + a_k^{\dagger} (iE_k) e^{iE_k x^0 - i\vec{k}\cdot\vec{x}} \right\}$$

$$[\phi(\vec{x}',t), \pi(\vec{x},t)] = \frac{1}{(2\pi)^3} \int \frac{d^3k'}{\sqrt{2E_{k'}}} \int \frac{d^3k}{\sqrt{2E_k}} \cdot \Big\{$$

$$\left\{ e^{-iE_k'x^0 + i\vec{k}'\cdot\vec{x}'} (-i) E_k e^{-iE_k x^0 + i\vec{k}\cdot\vec{x}} [a_{\vec{k}'}, a_{\vec{k}}] \right\}_{\text{kill}} +$$

$$\left\{ e^{-iE_k'x^0 + i\vec{k}'\cdot\vec{x}'} (iE_k) e^{iE_k x^0 - i\vec{k}\cdot\vec{x}} [a_{\vec{k}'}, a^{\dagger}_{\vec{k}}] \right\}_{\text{keep}} +$$

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$$+ \left\{ e^{iE_k'x^0 - i\vec{k}'\cdot\vec{x}'} e^{-iE_k x^0 + i\vec{k}\cdot\vec{x}} (-iE_k) [a^{\dagger}_{\vec{k}'}, a_{\vec{k}}] \right\}_{\text{keep}} +$$

$$\left\{ e^{iE_k'x^0 - i\vec{k}'\cdot\vec{x}'} (iE_k) e^{iE_k x^0 - i\vec{k}\cdot\vec{x}} [a^{\dagger}_{\vec{k}'}, a^{\dagger}_{\vec{k}}] \right\}_{\text{kill}} \Big\} =$$

$$= i\,\delta(\vec{x}-\vec{x}')$$

To do this we must require

$$[a_{\vec{k}'}, a_{\vec{k}}] = 0$$
$$[a^{\dagger}_{\vec{k}'}, a^{\dagger}_{\vec{k}}] = 0$$
$$[a_{\vec{k}'}, a^{\dagger}_{\vec{k}}] = \delta(\vec{k}'-\vec{k})$$

remember that $\delta(\vec{k}-\vec{k}') = \frac{1}{(2\pi)^3} \int d^3x\, e^{+i\vec{x}\cdot(\vec{k}-\vec{k}')}$

So
$$[\Phi(\vec{x}',t), \pi(\vec{x},t)] = \frac{1}{(2\pi)^3} \frac{i}{2} \int d^3k\, e^{i\vec{k}(\vec{x}'-\vec{x})} + \frac{1}{(2\pi)^3} \frac{i}{2} \int d^3k\, e^{i\vec{k}(\vec{x}'-\vec{x})}$$
$$= i\,\delta(\vec{x}-\vec{x}')$$

The Hamiltonian of the Scalar Field

The Hamiltonian of the scalar field (20)
is:

$$H = \int d^3x\, \Pi(\vec{x},t)\, \partial^0 \phi(\vec{x},t) - L$$

$$L = \int d^3x \left(\frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi - \frac{m^2}{2} \phi^2 \right)$$

$$H = \frac{1}{2} \int d^3x \left\{ \Pi^2 + (\vec{\nabla}\phi)^2 + m^2 \Phi^2 \right\}$$

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} \left[e^{i\vec{k}\vec{x} - iE_k t} a_{\vec{k}} + e^{-i\vec{k}\vec{x} + iE_k t} a_k^\dagger \right]$$

$$\Pi(x) = \frac{-i}{(2\pi)^{3/2}} \int d^3k \sqrt{\frac{E_k}{2}} \left[e^{i\vec{k}\cdot\vec{x} - iE_k t} a_{\vec{k}} - e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right]$$

$$\vec{\nabla}\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} (i\vec{k}) \left[e^{i\vec{k}\cdot\vec{x} - iE_k t} a_{\vec{k}} - e^{-i\vec{k}\vec{x} + iE_k t} a_k^\dagger \right]$$

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$$H = \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3x \Big\{ (-i)^2 \int d^3k \sqrt{\frac{E_k}{2}} \int d^3k' \sqrt{\frac{E_{k'}}{2}} \times$$

$$\Big\{ \left(e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k - e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right) \cdot \left(e^{i\vec{k}'\cdot\vec{x} - iE_{k'} t} a_{k'} - e^{-i\vec{k}'\cdot\vec{x} + iE_{k'} t} a_{k'}^\dagger \right) \Big\} +$$

$$+ i^2 \int \frac{d^3k}{\sqrt{2E_k}} \int \frac{d^3k'}{\sqrt{2E_{k'}}} (\vec{k}\cdot\vec{k}') \times$$

$$\Big\{ \left(e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k - e^{-i\vec{k}\cdot\vec{x} + iE_k t} a_k^\dagger \right) \cdot \left(e^{i\vec{k}'\cdot\vec{x} - iE_{k'} t} a_{k'} - e^{-i\vec{k}'\vec{x} + iE_{k'} t} a_{k'}^\dagger \right) \Big\} +$$

$$+ m^2 \int \frac{d^3k}{\sqrt{2E_k}} \int \frac{d^3k'}{\sqrt{2E_{k'}}} \left(e^{i\vec{k}\cdot\vec{x} - iE_k t} a_k + e^{-i\vec{k}\vec{x} + iE_k t} a_k^\dagger \right) \cdot \left(e^{i\vec{k}'\cdot\vec{x} - iE_{k'} t} a_{k'} + e^{-i\vec{k}'\cdot\vec{x} + iE_{k'} t} a_{k'}^\dagger \right)$$

The Hamiltonian with Creation and annihilation operators

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$\Rightarrow$

$$H = \frac{1}{2}\times\frac{1}{2}\int d^3k \Big\{ a_{\vec k} a_{-\vec k} \underbrace{\left[i^2 E_k - i^2 \frac{\vec k^2}{E_k} + \frac{m^2}{E_k} \right]}_{-E_k + \frac{k^2+m^2}{E_k} = 0} e^{2iE_k t} +$$

$$- a_{\vec k} a_{\vec k}^{\dagger} \underbrace{\left[(-i)^2 E_k + i^2 \frac{\vec k^2}{E_k} - \frac{m^2}{E_k} \right]}_{-E_k - \frac{k^2+m^2}{E_k} = -2E_k} +$$

$$- a_{\vec k}^{\dagger} a_{\vec k} \underbrace{\left[(-i)^2 E_k + i^2 \frac{\vec k^2}{E_k} - \frac{m^2}{E_k} \right]}_{-E_k - \frac{k^2+m^2}{E_k} = -2E_k}$$

$$+ a_{\vec k}^{\dagger} a_{-\vec k}^{\dagger} \underbrace{\left[(-i)^2 E_k - 1 i^2 \frac{\vec k^2}{E_k} + \frac{m^2}{E_k} \right]}_{-E_k + \frac{\vec k^2 + m^2}{E_k} = 0}$$

$$H = \frac{1}{4}\int d^3k \left\{ 2E_k a_{\vec k} a_{\vec k}^{\dagger} + 2E_k a_k^{\dagger} a_k \right\}$$

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$$H = \frac{1}{4}\int d^3k \left\{ 2E_k a_k a_k^{\dagger} + 2E_k a_k^{\dagger} a_k \right\}$$

$$H = \frac{1}{2}\int d^3k \left\{ a_{\vec k} a_{\vec k}^{\dagger} + a_{\vec k}^{\dagger} a_k \right\} E_k$$

$$[a_{\vec k}, a_{\vec k'}^{\dagger}] = \delta(\vec k - \vec k')$$

So $$H = \frac{1}{2}\int d^3k \, E_{\vec k} \left(2 a_{\vec k}^{\dagger} a_k + \delta(0) \right)$$

$$H = \frac{1}{2} 2 \int d^3k \, E_{\vec k} a_{\vec k}^{\dagger} a_k + \underbrace{\frac{1}{2}\int d^3k \, E_k \delta(0)}_{\text{the infinite sum of the } \hbar\omega \text{ of all H.O.}}$$

So

$$H = \int d^3k \, E_{\vec k} a_{\vec k}^{\dagger} a_{\vec k}$$

NOT MEASURED

So I GNORE

$$H|s\rangle = E_s|s\rangle$$

$$H a_k|s\rangle = \int d^3k' E_{k'} a_{k'}^{\dagger} a_{k'} a_k|s\rangle = \int d^3k' E_k' (a_k a_{k'}^{\dagger} - \delta(k-k'))|s\rangle$$

$$= \int d^3k' E_{k'} a_k a_{k'}^{\dagger} a_{k'} |s\rangle - E_k|s\rangle$$

$$= a_k H|s\rangle - E_k a_k|s\rangle$$

$$= (E_s - E_k) a_k|s\rangle$$ It lowers the energy by a quantum E_k

Summary: Quantum Field Theory

SUMMARY (24)

1. A Field theory is defined by a Lagrangian which is a function of the fields and the derivatives of the fields

2. $\delta S = 0$ gives the field equations

3. $[\pi(\vec{x},t), \phi(\vec{x}',t)] = -i\delta(\vec{x}-\vec{x}')$ type of commutation relations, when imposed on the solutions of the field equations result to a Quantum Field Theory that is

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega_k}}\left(e^{-ik\cdot x} a_{\vec{k}} + e^{ik\cdot x} a_{\vec{k}}^{+}\right)$$

with $[a_{\vec{k}}, a_{\vec{k}'}] = [a_{\vec{k}}^{+}, a_{\vec{k}'}^{+}] = 0$

$$[a_{\vec{k}}, a_{\vec{k}'}^{+}] = \delta(\vec{k}-\vec{k}')$$

4. But how do we get from all these the Feynman rules and the Cross-Section Calculations?? ⇒ Next time

? ? ?

Common to Classical and Quantum Field Theories

Commutation relations result to Quantum Field Theories