Symmetries and Conservation Laws

Symmetries and Conservation Laws (I)

Consider the Poincare transformation:

$$x^{\mu'} = x^\mu + \omega^{\mu\nu} x_\nu + a^\mu \quad \text{or}$$

$$\delta x^\mu = \omega^{\mu\nu} x_\nu + a^\mu \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

In general a field which could be a scalar or a vector or a spinor would transform as:

$$\Phi^{a'}(x') = \left[\delta^a{}_\beta - \tfrac{1}{2}\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\right]\Phi^\beta(x')$$

If the field is scalar then $\Sigma^a{}_\beta{}^{\mu\nu} = 0$

a, β are "field indices" which could be space time indices in the case of a vector field, but they could also be $a, b = 1,2,3,4$ in the case of a spinor.

$$\delta\Phi^a = \Phi^a(x') - \tfrac{1}{2}\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x') - \Phi^a(x)$$

$$\delta\Phi^a = \Phi^a(x+\omega x+a) - \Phi^a(x) - 1/2\,\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x')$$

$$\delta\Phi^a = a^\mu\partial_\mu\Phi^a(x) + \omega^{\mu\nu}x_\nu\partial_\mu\Phi^a(x) - \tfrac{1}{2}\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x)$$

(higher order terms dropped)

So:

$$\boxed{\delta\Phi^a = a^\mu\partial_\mu\Phi^a(x) + \tfrac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\Phi^a(x) + - 1/2\,\omega_{\mu\nu}\Sigma^a{}_\beta{}^{\mu\nu}\Phi^\beta(x)}$$

(II)

Under Poincare transformations the Lagrangian $\mathcal{L}$ should be a scalar that is

$$\mathcal{L}\left(\phi'(x'), \frac{\partial\phi'(x')}{\partial x_\mu}\right) = \mathcal{L}\left(\phi'(x'), \frac{\partial\Phi'(x')}{x'_\mu}\right)$$

(functional form remains the same)

Therefore:

$$\delta x^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\delta(\partial_\mu\phi^a) + \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\delta\phi^a\right] - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\right]\delta\phi^a + \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a$$

by Lagrange equation $= 0$

$$\therefore\ \delta x^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\delta\phi^a\right\}$$

"Emmy Noether"

$$(\omega^{\mu\nu}x_\nu + a^\mu)\partial_\mu\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\left(a^\rho\partial_\rho\phi^a + \tfrac{1}{2}\omega^{\rho\sigma}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi^a\right.\right.$$

$$\left.\left. - 1/2\,\omega_{\rho\sigma}\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\right\}\right. \Rightarrow$$

The Energy Momentum Tensor

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$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}a^\rho\partial_\rho\phi^a\right]-a^\mu\partial_\mu\mathcal{L}-\omega^{\mu\nu}x_\nu\partial_\mu\mathcal{L}+$$

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\frac{1}{2}\omega^{\rho\sigma}(x_\sigma\partial_\rho-x_\rho\partial_\sigma)\phi^a\right]+$$

$$-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\left(\frac{1}{2}\omega_{\rho\sigma}\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\right)\right]=0$$

$$a^\rho\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial_\rho\Phi^a-g^\mu{}_\rho\mathcal{L}\right]-\frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu-x_\mu\partial_\nu)\mathcal{L}$$

$$+\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\frac{1}{2}\omega^{\rho\sigma}(x_\sigma\partial_\rho-x_\rho\partial_\sigma)\phi^a\right]+$$

$$-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\frac{1}{2}\omega_{\rho\sigma}\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\right]=0$$

But $\boxed{T^{\mu\rho}=\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial^\rho\phi^a-g^{\mu\rho}\mathcal{L}}$ (Energy momentum tensor)

and $f^{\mu\rho\sigma}=1/2\,\Sigma^a{}_\beta{}^{\rho\sigma}\phi^\beta\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}$ (some tensor)

Therefore:

$$a_\rho\partial_\mu T^{\mu\rho}-\frac{1}{2}\omega^{\mu\nu}\partial_\rho\left[\delta^\rho{}_\mu x_\nu\mathcal{L}-x_\mu\delta^\rho{}_\nu\mathcal{L}\right]+$$

$$-1/2\,\omega_{\rho\sigma}\partial_\mu\left(f^{\mu\rho\sigma}-f^{\mu\sigma\rho}\right)+\frac{1}{2}\omega^{\rho\sigma}\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(x_\sigma\partial_\rho-x_\rho\partial_\sigma)\phi^a\right\}=0$$

Rename indices:

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$$a_\rho\partial_\mu T^{\mu\rho}-\frac{1}{2}\omega^{\rho\sigma}\partial_\mu\left[\delta^\mu{}_\rho x_\sigma\mathcal{L}-x_\rho\delta^\mu{}_\sigma\mathcal{L}\right]+$$

$$-\frac{1}{2}\omega_{\rho\sigma}\partial_\mu\left(f^{\mu\rho\sigma}-f^{\mu\sigma\rho}\right)+$$

$$1/2\,\omega^{\rho\sigma}\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(x_\sigma\partial_\rho-x_\rho\partial_\sigma)\phi^a\right]=0$$

$\Rightarrow$

$$a_\rho\partial_\mu T^{\mu\rho}+\frac{1}{2}\omega^{\rho\sigma}\partial_\mu\Big\{x_\rho\delta^\mu{}_\sigma\mathcal{L}-x_\sigma\delta^\mu{}_\rho\mathcal{L}+$$

$$f^{\mu\sigma\rho}-f^{\mu\rho\sigma}+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(x_\sigma\partial_\rho-x_\rho\partial_\sigma)\phi^a\Big\}=0$$

Define
$$\mathbb{M}^\mu{}_{\rho\sigma}=f^{\mu\sigma\rho}-f^{\mu\rho\sigma}+$$
$$+x_\sigma\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial_\rho\phi^a-\delta^\mu{}_\rho\mathcal{L}\right)$$
$$-x_\rho\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\partial_\sigma\phi^a-\delta^\mu{}_\sigma\mathcal{L}\right)$$

That is $\boxed{\mathbb{M}^\mu{}_{\rho\sigma}=f^{\mu\sigma\rho}-f^{\mu\rho\sigma}+x_\sigma T^\mu{}_\rho-x_\rho T^\mu{}_\sigma}$

Internal Symmetries of the Lagrangian

In Conclusion: (V)

$$a_\rho \partial_\mu T^{\mu\rho} + \frac{1}{2}\omega^{\rho\sigma}\partial_\mu M^{\mu}{}_{\rho\sigma} = 0$$

If this is true for any a_ρ and $\omega_{\rho\sigma}$ (any Poincare transformation) then it must be:

$$\partial_\mu T^{\mu\rho} = 0 \quad \text{Conservation of Energy-Momentum}$$

$$\partial_\mu M^{\mu}{}_{\rho\sigma} = 0 \quad \text{Angular momentum is conserved}$$

$$T^{\mu\rho} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}\partial^\rho \phi^a - g^{\mu\rho}\mathcal{L}$$

$$f^{\mu\rho\sigma} = \frac{1}{2}\Sigma^{a}{}_{\beta}{}^{\rho\sigma}\phi^\beta \cdot \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}$$

$$M^{\mu\rho\sigma} = f^{\mu\sigma\rho} - f^{\mu\rho\sigma} + x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma}$$

And

$$\Sigma^{a}{}_{\beta}{}^{\rho\sigma} = \begin{cases} 0 & \text{for scalar fields} \\ g^{a\rho}g_\beta{}^\sigma - g^{a\sigma}g_\beta{}^\rho & \text{for vector fields} \\ 1/4\,[\gamma^\rho, \gamma^\sigma]^a{}_\beta & \text{Spinors} \end{cases}$$

INTERNAL SYMMETRIES (VI)

Often a Lagrangian is invariant under transformations of the type:

$$\Phi(x) \to e^{-i\epsilon\lambda}\Phi(x) \cong 1 - i\epsilon\lambda\cdot\Phi$$

Φ in general is $\begin{pmatrix}\phi_1\\ \phi_2 \\ \vdots \\ \phi_n\end{pmatrix}$ and λ is $n\times n$ matrix:

$$\Phi_i(x) \to \Phi_i(x) - i\epsilon\lambda_{ij}\Phi_j(x)$$

The transformations form a group and λ_{ij} are the group generators

If one constructs a Lagrangian which is invariant under these transformations then we have

$$\delta\mathcal{L} = 0 = \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta(\partial_\mu \phi_i)$$

$$= \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\phi_i\right] + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\right)\delta\phi_i}_{\overset{\shortparallel}{0} \text{ using Lagrange Equations}}$$

Enough for today........

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So

$$\delta\mathcal{L}=0=\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right\}$$

$$\boxed{\partial_\mu J^\mu=0}$$

$$J^\mu=\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i \Rightarrow$$

$$J^\mu=\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}(-i\epsilon)\lambda_{ij}\phi_j(x)$$

$$\boxed{J^\mu=-i\epsilon\,\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\lambda_{ij}\phi_j(x)}$$

So internal symmetries lead to conserved currents:

$$\partial_0 J^0=0 \Rightarrow J^0=-i\epsilon\frac{\partial\mathcal{L}}{\partial\partial_0\phi_i}\lambda_{ij}\phi_j$$

$$J^0=-i\epsilon\,\pi_i\lambda_{ij}\phi_j$$

$$Q=-i\int d^3x\,\pi_i\lambda_{ij}\phi_j$$

$$dQ/dt=0$$