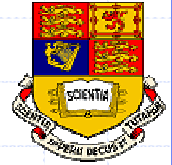


The Dirac Equation and Solutions



DIRAC EQUATION SOLUTIONS (BRIEF SUMMARY)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \text{Let } \psi = U e^{-ipx} \Rightarrow \quad \textcircled{1}$$

$$(\not{p} - m)\psi = 0 \Rightarrow (\gamma^0 p^0 - \vec{\gamma} \cdot \vec{p} - m)U = 0 \Rightarrow$$

$$(\vec{\gamma} \cdot \vec{p} + m)U = \gamma^0 p^0 U \Rightarrow (\gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 m)U = p^0 U$$

$$\Rightarrow \underbrace{\left(\vec{\alpha} \cdot \vec{p} + \beta m \right)}_H U = p^0 U; \text{ where } \alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \textcircled{I} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$\Rightarrow H \cdot U = p^0 U$. Need to solve this equation

$$\textcircled{I} \Rightarrow \begin{bmatrix} m - p^0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -(m + p^0) \end{bmatrix} U = 0 \quad \text{if } U \neq 0 \quad \textcircled{II}$$

$$\Rightarrow \det[\] = 0 \Rightarrow -(m - p^0)(m + p^0) - \vec{p}^2 = 0$$

$$\therefore p^0^2 = \vec{p}^2 + m^2 \quad \text{and } p^0 \text{ can be identified with Energy}$$

$$\therefore p^0 = E = \pm \sqrt{\vec{p}^2 + m^2}$$

In other words we have both positive and negative energy solutions just like in the Klein Gordon equation.

$$\textcircled{II} \begin{bmatrix} m - p^0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -(m + p^0) \end{bmatrix} \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = 0 \Rightarrow \begin{aligned} \chi &= \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \varphi \quad \textcircled{A} \\ \varphi &= \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi \quad \textcircled{B} \end{aligned}$$

\textcircled{B} is well defined for $E > 0$ and \textcircled{A} is well defined for $E < 0$

So for positive energy solutions we have: $\textcircled{2}$

$$\psi = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \end{pmatrix} \chi^\pm e^{-ipx} \quad (p^0 = E > 0)$$

and for negative energy solutions we have

$$\psi = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \\ 1 \end{pmatrix} \chi^\pm e^{-ipx} \quad (p^0 = E \leq 0)$$

where $\chi^\pm = (\uparrow), (\downarrow)$

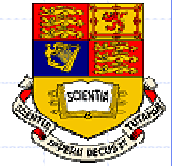
How ever one could have tried $\psi = U e^{+ipx}$ at the beginning. In this case the solutions would have been

$$\psi = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \\ 1 \end{pmatrix} \chi^\pm e^{+ipx} \quad (E = p^0 > 0)$$

$$\psi = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \end{pmatrix} \chi^\pm e^{+ipx} \quad (E = p^0 < 0)$$

The two sets are related by $\begin{aligned} E &\rightarrow E' = -E \\ \vec{p} &\rightarrow \vec{p}' = -\vec{p} \end{aligned}$

Dirac Equation: Comments



OBSERVATIONS

I. $[H, \vec{L} + \frac{1}{2} \vec{\Sigma}] = 0$ where

$$H = \vec{\alpha} \cdot \vec{p} + \beta m ; \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

\vec{L} = ORBITAL ANGULAR MOMENTUM

IN OTHER WORDS: IT IS THE TOTAL $\vec{J}_{TOT} = \vec{L} + \frac{1}{2} \vec{\Sigma}$ THAT IS CONSERVED. THE DIRAC EQ. REQUIRES / DESCRIBES $s=1/2$ PARTICLES. EACH SOLUTION IS TWO-FOLD DEGENERATE DUE TO SPIN.

II. NEGATIVE ENERGY SOLUTIONS ARE ALSO PREDICTED JUST LIKE IN THE KLEIN-GORDON CASE AND NEED TO BE INTERPRETED.

$$\text{III. } \left. \begin{aligned} i\gamma^{\mu} \partial_{\mu} \psi - m\psi = 0 &\Rightarrow \bar{\psi} (i\gamma^{\mu} \partial_{\mu} - m)\psi = 0 \\ \text{and } \bar{\psi} (i\gamma^{\mu} \partial_{\mu} + m) = 0 &\Rightarrow \bar{\psi} (i\gamma^{\mu} \partial_{\mu} + m)\psi = 0 \end{aligned} \right\}$$

$$i\bar{\psi} \gamma^{\mu} \partial_{\mu} \psi + i\partial_{\mu} \bar{\psi} \gamma^{\mu} \psi = 0 \Rightarrow$$

$$\partial_{\mu} (\bar{\psi} \gamma^{\mu} \psi) = 0$$

THERE IS A CONSERVED 4-VECTOR

$$\boxed{J^{\mu} = \bar{\psi} \gamma^{\mu} \psi}$$

$$\rho = \bar{\psi} \gamma^0 \psi = \psi^{\dagger} \gamma^0 \psi = \psi^{\dagger} \psi \text{ is Prob. density}$$

$$\vec{J} = \bar{\psi} \vec{\gamma} \psi \text{ 3-VECTOR CURRENT}$$

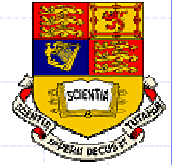
$$\text{For } E > 0 \quad \rho = |N|^2 \frac{2E}{E+m} \gg 0$$

$$\text{For } E < 0 \quad \rho = |N|^2 \frac{2E}{E-m} = |N|^2 \frac{2|E|}{|E+m|} \gg 0$$

BOTH $E > 0$
AND $E < 0$
SOLUTIONS
HAVE $\rho > 0$

③ This is a difference between the Klein-Gordon equation where negative energy solutions were associated with negative probability. The Dirac equation predicts always positive probabilities for both positive and negative energy solutions. ④

IV The Dirac equation also predicts the correct magnetic moment of the electron which corresponds to a gyromagnetic ratio of $g=2$



Covariance of the Dirac Equation

LORENTZ COVARIANCE OF THE DIRAC EQUATION (5)

Recall that a Lorentz transformation from frame $O \rightarrow O'$ is defined as

$$X^{M'} = \Lambda^M_{\nu} X^{\nu} \quad \text{with}$$

$$\Lambda^M_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(if it is a boost at the x-direction)

$$\text{Also } \delta x^{M'} = \Lambda^M_{\nu} \delta x^{\nu} \Rightarrow \frac{\partial x^{M'}}{\partial x^{\nu}} = \Lambda^M_{\nu}$$

$$\text{and } \delta x^{M'} = \frac{\partial x^{M'}}{\partial x^{\nu}} \delta x^{\nu} \Rightarrow \frac{\partial}{\partial x^{\nu}} = \frac{\partial x^{M'}}{\partial x^{\nu}} \frac{\partial}{\partial x^{M'}} \Rightarrow$$

$$\frac{\partial}{\partial x^{\nu}} = \Lambda^M_{\nu} \frac{\partial}{\partial x^{M'}} \quad (3)$$

Suppose that in O we have that

$$(i\gamma^M \partial_M - m)\psi(x) = 0$$

and in O' :

$$(i\gamma^{M'} \partial_{M'} - m)\psi'(x') = 0$$

IF THE DIRAC EQUATION IS COVARIANT (6)
TWO CONDITIONS MUST BE TRUE:

I. WE HAVE A METHOD (TRANSFORMATION) WHICH BRINGS $\psi(x) \rightarrow \psi'(x')$

II. THE DIRAC EQUATION IS INVARIANT UNDER THIS CHANGE

Suppose $\psi'(x') = \psi(\Lambda \cdot x) = S(\Lambda)\psi(x)$
which means that $\psi(x) = S^{-1}(\Lambda)\psi'(x')$ (7)

START FROM REF. FRAME O :

$$(i\gamma^M \partial_M - m)\psi(x) = 0 \quad (7)$$

$$(i\gamma^M \partial_M - m)S^{-1}(\Lambda)\psi'(x') = 0 \Rightarrow$$

$$(iS(\Lambda)\gamma^M S^{-1}(\Lambda) \frac{\partial}{\partial x^M} - m)\psi'(x') = 0 \quad (8)$$

$$(iS(\Lambda)\gamma^M S^{-1}(\Lambda) \Lambda^{\nu}_M \frac{\partial}{\partial x^{\nu}} - m)\psi'(x') = 0$$

SO IF THE EQUATION IS TO REMAIN INVARIANT FROM FRAME TO FRAME THEN $S(\Lambda)$ MUST SATISFY:

$$S(\Lambda)\gamma^M S^{-1}(\Lambda) \Lambda^{\nu}_M = \gamma^{\nu}$$



Parity for Spin = 1/2 fields

Such a matrix exists and is defined by

$$S(\Lambda) = e^{-\frac{i}{4} \sigma_{\mu\nu} \Delta\omega^{\mu\nu}}$$

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu \gamma_\nu]$$

$$\Lambda^M{}_\nu = \delta^M{}_\nu + \Delta\omega^M{}_\nu$$

(INFINITECIMAL LORENTZ?)

$S(\Lambda)$ has the property that

$$S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda) \gamma^0$$

Let's study now the parity properties of the Dirac equation: (B)

Suppose that \mathcal{O} and \mathcal{O}' are parity related frames. In other words

$$\begin{aligned} t &\xrightarrow{P} t' = t \\ \vec{x} &\xrightarrow{P} \vec{x}' = -\vec{x} \end{aligned}$$

$$(i\gamma^\mu \partial'_\mu - m) \psi'(x') = 0 \quad \text{in the frame } \mathcal{O}' \Rightarrow$$

$$(i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \vec{\nabla}' - m) \psi'(x') = 0 \Rightarrow$$

$$(i\gamma^0 \partial_0 + i\gamma^i \partial_i - m) \psi'(x') = 0 \Rightarrow$$

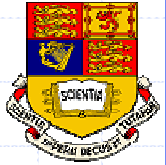
$$(i\gamma^0 \partial_0 - i\vec{\gamma} \cdot \vec{\nabla}' - m) \gamma^0 \psi'(x') = 0 \Rightarrow$$

$$(i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \vec{\nabla} - m) \gamma^0 \psi'(x') = 0$$

$$(i\gamma^\mu \partial_\mu - m) \gamma^0 \psi'(x') = 0$$

$$\therefore \psi'(x') = \gamma^0 \psi(x) \Rightarrow P\psi(x) = \psi'(x) = \gamma^0 \psi(x)$$

Parity for Positive and Negative Energy Solutions



So the Dirac equation is invariant under (9) parity provided that

$$\psi(x) \rightarrow P\psi(x) = \psi'(x') = \gamma^0 \psi(x)$$

From this we see that POSITIVE AND NEGATIVE ENERGY SOLUTIONS HAVE OPPOSITE PARITY !!!

To see this, go to the particle rest frame and

$$\left. \begin{aligned} P \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= \gamma^0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (+1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ P \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} &= \gamma^0 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = (+1) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \right\} m > 0$$

$$\left. \begin{aligned} P \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} &= \gamma^0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = (-1) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ P \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} &= \gamma^0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \right\} m < 0$$