The Dirac Equation and Solutions

DIRAC EQUATION SOLUTIONS (BRIEF SUMMARY) ①

$(i\gamma^\mu\partial_\mu - m)\psi = 0$ Let $\psi = u e^{-ipx} \Rightarrow$

$(\not{p} - m)\psi = 0 \Rightarrow (\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} - m)u = 0 \Rightarrow$

$(\vec{\gamma}\cdot\vec{p} + m)u = \gamma^0 p^0 u \Rightarrow (\gamma^0\vec{\gamma}\cdot\vec{p} + \gamma^0 m)u = p^0 u$

$\Rightarrow \underbrace{(\vec{\alpha}\cdot\vec{p} + \beta m)}_{H} u = p^0 u$; where $\alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ (I)

$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$\Rightarrow H\cdot u = p^0 u$. Need to solve this equation

(I) $\Rightarrow \begin{bmatrix} m - p^0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(m+p^0) \end{bmatrix} u = 0$ if $u \neq 0$ (II)

$\Rightarrow \det[\ ] = 0 \Rightarrow -(m-p^0)(m+p^0) - \vec{p}^{\,2} = 0$

$\therefore p^{0\,2} = \vec{p}^{\,2} + m^2$ and p^0 can be identified with Energy

$\therefore p^0 = E = \pm\sqrt{\vec{p}^{\,2} + m^2}$

In other words we have both positive and negative energy solutions just like in the Klein Gordon equation.

(II) $\begin{bmatrix} m - p^0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(m+p^0) \end{bmatrix}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} = 0 \Rightarrow$ $\chi = \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M}\varphi$ (A)

$\varphi = \dfrac{\vec{\sigma}\cdot\vec{p}}{E+M}\chi$ (B)

(B) is well defined for $E>0$ and (A) is well defined for $E<0$

So for positive energy solutions we have: ②

$$\psi = N\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix}\chi^{\pm} e^{-ip\cdot x} \quad (p^0 = E \geq 0)$$

and for negative energy solutions we have

$$\psi = N\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ 1 \end{pmatrix}\chi^{\pm} e^{-ipx} \quad (p^0 = E \leq 0)$$

where $\chi^{\pm} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

However one could have tried $\psi = v e^{+ipx}$ at the beginning. In this case the solutions would have been:

$$\psi = N\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+M} \\ 1 \end{pmatrix}\chi^{\pm} e^{+ipx} \quad (E = p^0 > 0)$$

$$\psi = N\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \end{pmatrix}\chi^{\pm} e^{+ipx} \quad (E = p^0 < 0)$$

The two sets are related by $E \to E' = -E$, $\vec{p} \to \vec{p}\,' = -\vec{p}$

Dirac Equation: Comments

OBSERVATIONS ③

I. $[H, \vec{L} + \frac{1}{2}\vec{\Sigma}] = 0$ where

$$H = \vec{\alpha}\cdot\vec{p} + \beta m \; ; \; \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$\vec{L}$ = ORBITAL ANGULAR MOMENTUM

IN OTHER WORDS: IT IS THE TOTAL $\vec{J}_{TOT} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ THAT IS CONSERVED. THE DIRAC EQ. REQUIRES/DESCRIBES $S=1/2$ PARTICLES. EACH SOLUTION IS TWO-FOLD DEGENERATE DUE TO SPIN.

II. NEGATIVE ENERGY SOLUTIONS ARE ALSO PREDICTED JUST LIKE IN THE KLEIN-GORDON CASE AND NEED TO BE INTERPRETED.

III. $i\gamma^\mu \partial_\mu \psi - m\psi = 0 \Rightarrow \bar{\psi}(i\gamma^\mu \overrightarrow{\partial_\mu} - m)\psi = 0$

and $\bar{\psi}(i\gamma^\mu \overleftarrow{\partial_\mu} + m) = 0 \Rightarrow \bar{\psi}(i\gamma^\mu \overleftarrow{\partial_\mu} + m)\psi = 0$

$$i\bar{\psi}\gamma^\mu \partial_\mu \psi + i\partial_\mu \bar{\psi}\gamma^\mu \psi = 0 \Rightarrow$$

$$\partial_\mu(\bar{\psi}\gamma^\mu \psi) = 0$$

THERE IS A CONSERVED 4-VECTOR $\boxed{J^\mu = \bar{\psi}\gamma^\mu\psi}$

$\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$ is Prob. density

$\vec{J} = \bar{\psi}\vec{\gamma}\psi$ 3-VECTOR CURRENT

For $E>0$ $\quad \rho = |N|^2 \frac{2E}{E+m} \geq 0$

For $E<0$ $\quad \rho = |N|^2 \frac{2E}{E-m} = |N|^2 \frac{2|E|}{|E|+m} \geq 0$

BOTH $E>0$ AND $E<0$ SOLUTIONS HAVE $\rho>0$

This is a difference between the Klein-Gordon equation where negative energy solutions where associated with negative probability. The Dirac equation predicts always positive probabilities for both positive and negative energy solutions. ④

IV. The Dirac equation also predicts the correct magnetic moment of the electron which corresponds to a gyromagnetic ratio of $g=2$

Covariance of the Dirac Equation

LORENTZ COVARIANCE OF THE DIRAC EQUATION (5)

Recall that a Lorentz transformation from frame $O \to O'$ is defined as

$$\boxed{x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}}$$ with

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(if it is a boost at the x-direction)

Also $\delta x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}\, \delta x^{\nu} \Rightarrow \dfrac{\partial x^{\mu\prime}}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\nu}$

and $\delta x^{\mu\prime} = \dfrac{\partial x^{\mu\prime}}{\partial x^{\nu}} \delta x^{\nu} \Rightarrow \dfrac{\partial}{\partial x^{\mu\prime}} = \dfrac{\partial x^{\nu}}{\partial x^{\mu\prime}} \dfrac{\partial}{\partial x^{\nu}} \Rightarrow$

$$\boxed{\frac{\partial}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\nu} \frac{\partial}{\partial x^{\mu\prime}}} \quad (1)$$

Suppose that in O we have that

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0$$

and in O':

$$(i\gamma^{\mu}\partial'_{\mu} - m)\psi'(x') = 0$$

(6)

IF THE DIRAC EQUATION IS COVARIANT TWO CONDITIONS MUST BE TRUE:

I. WE HAVE A METHOD (TRANSFORMATION) WHICH BRINGS $\psi(x) \to \psi'(x')$

II. THE DIRAC EQUATION IS INVARIANT UNDER THIS CHANGE

Suppose $\psi'(x') = \psi'(\Lambda \cdot x) = S(\Lambda)\psi(x)$

which means that $\boxed{\psi(x) = S^{-1}(\Lambda)\,\psi'(x')}$ (T)

START FROM REF. FRAME O:

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \overset{(T)}{\Longrightarrow}$$

$$(i\gamma^{\mu}\partial_{\mu} - m)\, S^{-1}(\Lambda)\psi'(x') = 0 \Rightarrow$$

$$\left(i\, S(\Lambda)\gamma^{\mu} S^{-1}(\Lambda) \frac{\partial}{\partial x^{\mu}} - m\right)\psi'(x') = 0 \overset{(1)}{\Rightarrow}$$

$$\boxed{\left(i\, S(\Lambda)\gamma^{\mu} S^{-1}(\Lambda)\, \Lambda^{\nu}{}_{\mu} \frac{\partial}{\partial x^{\nu\prime}} - m\right)\psi'(x') = 0}$$

SO IF THE EQUATION IS TO REMAIN INVARIANT FROM FRAME TO FRAME THEN $S(\Lambda)$ MUST SATISFY:

$$\boxed{S(\Lambda)\gamma^{\mu} S^{-1}(\Lambda)\, \Lambda^{\nu}{}_{\mu} = \gamma^{\nu}}$$

Parity for Spin = 1/2 fields

(7)

Such a matrix exists and is defined by

$$S(\Lambda) = e^{-\frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu}}$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu \gamma_\nu]$$

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \Delta\omega^\mu{}_\nu$$

(INFINITECIMAL LORENTZ)

$S(\Lambda)$ has the property that

$$\boxed{S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda)\gamma^0}$$

(8)

Lets study now the parity properties of the Dirac equation:

Suppose that O and O' are parity related frames. In other words

$$t \xrightarrow{\mathbb{P}} t' = t$$
$$\vec{x} \xrightarrow{\mathbb{P}} \vec{x}' = -\vec{x}$$

$(i\gamma^\mu\partial'_\mu - m)\psi'(x') = 0$ in the frame O' $\Rightarrow$

$$(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\vec{\nabla}' - m)\psi'(x') = 0 \Rightarrow$$

$$(i\gamma^0\gamma^0\partial_0 + i\gamma^0\vec{\gamma}\cdot\vec{\nabla}' - m\gamma^0)\psi'(x') = 0 \Rightarrow$$

$(-)$

$$(i\gamma^0\partial_0 - i\vec{\gamma}\vec{\nabla}' - m)\gamma^0\psi'(x') = 0 \Rightarrow$$

$$(i\gamma^0\partial_0 + i\vec{\gamma}\vec{\nabla} - m)\gamma^0\psi'(x') = 0$$

$$(i\gamma^\mu\partial_\mu - m)\gamma^0\psi'(x') = 0$$

$$\therefore \psi'(x') = \gamma^0\psi(x) \Rightarrow \mathbb{P}\psi(x) = \psi'(x') = \gamma^0\psi(x)$$

Parity for Positive and Negative Energy Solutions

So the Dirac equation is invariant under parity provided that (9)

$$\boxed{\psi(x) \longrightarrow \mathbb{P}\psi(x) = \psi'(x') = \gamma^0\psi(x)}$$

From this we see that POSITIVE AND NEGATIVE ENERGY SOLUTIONS HAVE OPPOSITE PARITY !!!

To see this, go to the particle rest frame and

$$\left.\begin{aligned} \mathbb{P}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} &= \gamma^0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = (+1)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \\ \mathbb{P}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} &= \gamma^0\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = (+1)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \end{aligned}\right] \quad m>0$$

$$\left.\begin{aligned} \mathbb{P}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} &= \gamma^0\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = (-1)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \\ \mathbb{P}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} &= \gamma^0\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = (-1)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{aligned}\right] \quad m<0$$