# Introduction to Covariant Notation

(2)

Lecture II: Quantum Field Theory (1)

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Review of covariant notations:

4-Vectors: A point in space-time can be defined by a 4-vector $x^\mu = (x^0, \vec{x})$ where $\mu = 0,1,2,3$ and $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$x^\mu(\tau) = (x^0(\tau), \vec{x}(\tau))$ describes a trajectory of an object in space time. The quantity $(x^0)^2 - \vec{x}^2$ is invariant under Lorentz transformations.

- Define CONTRAVARIANT 4-vectors as $x^\mu = (x^0, \vec{x})$ and COVARIANT 4-vectors as $x_\mu = (x^0, -\vec{x})$
- Define the dot product of two 4-vectors $a^\mu, b^\mu$ as $a^\mu b_\mu = a^0 b^0 - \vec{a}\cdot\vec{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$
  This product is invariant under Lorentz transformations
- Define a metric as $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
  You can think then of a dot product as

$$(a^0, a^1, a^2, a^3) \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

Using the metric one can see that the dot product can be written as

$$\underset{\substack{\uparrow \\ \text{Einstein's Convention}}}{a^\mu b_\mu} = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \vec{a}\cdot\vec{b}$$

"Any index, that appears twice once as a subscript and once as a superscript is understood to be summed"

- You can convert from covariant to contravariant vectors using the metric:

$$x^\mu = g^{\mu\nu} x_\nu \quad \text{or}$$
$$x_\mu = g_{\mu\nu} x^\nu$$

but $x^\mu x_\mu = g^{\mu\alpha} x_\alpha g_{\mu\beta} x^\beta \Rightarrow$

$$g^{\mu\alpha} g_{\mu\beta} = \delta^\alpha{}_\beta = \begin{cases} +1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

- Under Lorentz transformations, 4-vectors transform as:

$$\left.\begin{aligned} x'^\mu &= \Lambda^\mu{}_\nu x^\nu \\ x'_\mu &= \Lambda_\mu{}^\alpha x_\alpha \end{aligned}\right\} \Rightarrow x'_\mu \cdot x'^\mu = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta x_\alpha x^\beta$$

$$\Rightarrow \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \delta^\alpha{}_\beta$$

Therefore $\Lambda_\mu{}^\alpha$ is the inverse of $\Lambda^\mu{}_\beta$

# Maxwell's Equation in covariant notation

Derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)$$

and $\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \Box$

- So one can write the wave equation in a compact form as $\partial_\mu \partial^\mu \psi = 0$ or $\Box \psi = 0$
- Consider now the Maxwell equations

$$\nabla \cdot \vec{E} = 4\pi\rho$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

recall that:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Define a second-rank tensor which is antisymm and is defined as $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$A^\mu = (\Phi, \vec{A})$$

Under Lorentz $F'^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu}$

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So
$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

as before one can convert to $F_{\alpha\beta}$ as

$$F_{\alpha\beta} = g_{\alpha\gamma}\, g_{\beta\delta}\, F^{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

And
$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ &\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \end{aligned}\right\} \Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$$

also $\partial_\mu \partial_\nu F^{\mu\nu} = 0 = \frac{4\pi}{c}\partial_\nu J^\nu$

$$\therefore \boxed{\partial_\nu J^\nu = 0}$$

Define also $\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$

where
$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \alpha=0, \beta=1, \gamma=2, \delta=3 \oplus \text{even perm.} \\ -1 & \text{any odd perm.} \\ 0 & \text{if any to indices are equal} \end{cases}$$

$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{B} \\ &\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \end{aligned}\right\} \Rightarrow \boxed{\partial_\alpha \tilde{F}^{\alpha\beta} = 0}$$

# Action and Lagrangian in Field Theory

Classical Field Theory (5)

- Consider the action $S$

$$S = \int d^4x\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \quad \text{or}$$

$$S = \int dx^0\, L = \int dx^0 \int d^3x\, \mathcal{L}(\phi(x), \partial_\mu \phi(x))$$

- $\mathcal{L}$ is called Lagrangian density
- $L$ is the Lagrangian
- $\Phi(x)$ is a field which in Quantum Field theory will be an operator that can create or destroy a particle

The field equations for $\Phi(x)$ can be obtained from $S$ by requiring that $\delta S = 0 \Rightarrow S = S_{min}$ when one considers $\phi \to \phi' = \phi + \delta\phi$

The action $S$ must be:

(a) Lorentz invariant + invariant under translations that is Poincaré invariant

(b) function of the fields and their derivatives (translational invariance again)

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(c) depends on the fields taken at one space-time point $x^\mu$ only, leading to a local field theory

(d) $S$ must be a real # $\Rightarrow$ probability is conserved. An action which is complex leads to absorption i.e. matter disappears into nothing $\Rightarrow$ no good

(e) must lead to classical equations of motion that involve no-higher than second-order derivatives $\Rightarrow$ we want a theory that has causal solutions

- If $S$ is a number then

$$d^4x \sim L^4 \sim m^{-4} \Rightarrow \mathcal{L} \sim L^{-4} \sim m^4$$

So a Lagrangian density of the form

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2$$

is a good guess

# The Lagrange Equations in Field Theory

Lagrange Equations in Field Theory (7)

$$\left.\begin{array}{l} S=\int d^4x\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\ \delta S = 0 \end{array}\right\} \Rightarrow$$

$$\int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \underbrace{\delta(\partial_\mu \phi)}_{\partial_\mu(\delta\phi)} \right\} = 0$$

$$\int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi\right) - \delta\phi\, \partial_\mu\left(\frac{\partial L}{\partial \partial_\mu \phi}\right)\right\} = 0$$

$$\int d^4x \left\{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right)\right\}\delta\phi + \underbrace{\int d^4x\, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right)\delta\phi}_{0} = 0$$

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$$\therefore \boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right) - \frac{\partial L}{\partial \phi} = 0}$$

Recall $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$ in classical mechanics

Example: Scalar field (8)

$$\left.\begin{array}{l} \mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 \\ \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\right) - \frac{\partial L}{\partial \phi} = 0 \end{array}\right\} \Rightarrow$$

$$\frac{\partial}{\partial x^\mu}\left\{\frac{1}{2}\delta^\mu_{\ \alpha}\,\partial^\alpha\phi + \frac{1}{2}\partial_\alpha\phi\, g^{\mu\alpha}\right\} + \frac{m^2}{2}2\phi = 0$$

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0$$

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0 \quad \text{Klein-Gordon Equation}$$

or $$(\Box + m^2)\phi(x) = 0$$

# The Vector Field Lagrangian

Example: The Electromagnetic Field

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Rightarrow \text{Lorentz scalar}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow F_{\mu\nu} = -F_{\nu\mu}$$

$$\mathcal{L} = \frac{1}{4}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) F^{\alpha\beta} = \frac{1}{4}(\partial_\alpha A_\beta F^{\alpha\beta} - \partial_\beta A_\alpha F^{\alpha\beta})$$

$$\mathcal{L} = \frac{1}{4}(\partial_\alpha A_\beta F^{\alpha\beta} + \partial_\beta A_\alpha F^{\beta\alpha}) = \frac{1}{2} \partial_\alpha A_\beta F^{\alpha\beta}$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = \frac{\partial}{\partial \partial_\mu A_\nu} \left\{ \frac{1}{2} \partial_\alpha A_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \right\}$$

$$= \left\{ \frac{1}{2} \delta^\mu{}_\alpha \delta^\nu{}_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + 1/2\, \partial_\alpha A_\beta (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) \right\}$$

$$= \frac{1}{2} \left\{ \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu A^\nu - \partial^\nu A^\mu \right\}$$

$$= F^{\mu\nu} \quad ; \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} = 0$$ Maxwell equation no current

⑨ Introduce the Field current coupling ⑩

$$\mathcal{L}_I = \frac{4\pi}{c} J_\mu A^\mu$$

(later it will come as a result of $U(1)$ local gauge theory)

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_I = +\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{4\pi}{c} J_\mu A^\mu$$

Then $\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{4\pi}{c} J_\alpha g^{\nu\alpha} = J^\nu \cdot \frac{4\pi}{c}$

$$\therefore \quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

# The Spin 1/2 Lagrangian and the Dirac Equation

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## Pauli Matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

PROPERTIES: $\sigma_i^2 = 1$

$$\sigma_i \sigma_j = \delta_{ij} + i\,\epsilon_{ijk}\sigma_k$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}$$

$\{1, \sigma_1, \sigma_2, \sigma_3\}$ is a basis in the 2x2 matrix space

## Dirac Matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^5\} = 0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

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## The Dirac Field

Define a 4-component spinor $\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

and $\bar{\psi} = \psi^\dagger \gamma^0$ that is

$$\bar{\psi}_\alpha = \psi^\dagger_\beta (\gamma^0)_{\beta\alpha}$$

Next consider the Lagrangian:

$$\mathcal{L}(x) = \frac{i}{2}\left\{ \bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi \right\} - m\bar{\psi}\psi$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}_\alpha} = \frac{-i}{2} (\gamma^\mu)_{\alpha\beta} \psi_\beta$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_\alpha} = \frac{i}{2} (\gamma^\mu)_{\alpha\beta} \partial_\mu \psi_\beta - m\psi_\alpha$$

$$\therefore \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}_\alpha} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}_\alpha} = 0 \Longrightarrow$$

$$\underbrace{-\frac{i}{2} \gamma^\mu_{\alpha\beta} \partial_\mu \psi_\beta + \frac{-i}{2} \gamma^\mu_{\alpha\beta} \partial_\mu \psi_\beta} + m\psi_\alpha = 0$$

$$-i\,\partial\!\!\!/\,\psi + m\psi = 0$$

$$(i\,\partial\!\!\!/ - m)\psi = 0 \quad \text{Dirac's Equation}$$

# The Dirac Equation

⑬

or $$\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_\alpha} = \frac{i}{2} \bar{\psi}_\beta \gamma^\mu_{\beta\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial \psi_\alpha} = -\frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu - m \bar{\psi}$$

$$\therefore \partial_\mu \left( \frac{i}{2} \bar{\psi} \gamma^\mu \right) + \frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0 \Rightarrow$$

$$i \partial_\mu \bar{\psi} \gamma^\mu + m \bar{\psi} = 0 \quad \Rightarrow$$

$$\bar{\psi} (i \overleftarrow{\not\partial} + m) = 0$$

just another way to write Dirac's equation for $\bar{\psi}$