Symmetries in Classical and Quantum Physics

- Symmetries in Classical Physics
  - Conservation of Energy
  - Conservation of Momentum
  - Conservation of Angular Momentum.
- Symmetries in quantum mechanics.
- Example of symmetry in particle physics
  - Isospin in strong interactions

Lagrange Equations in Classical Mechanics

Classical Mechanics

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10.02

The principle of least action or
HAMILTON'S PRINCIPLE:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = \text{action} \sim [E]\cdot[T]$$

L = lagrangian $\sim [E]$

Equations of motion ($q = q(t)$) are derived by requiring that $S = S_{min} \Rightarrow \delta S = 0$

Consider path variations where $q(t) \to q'(t) = q(t) + \delta q(t)$ and $\delta q_i(t_1) = \delta q_i(t_2) = 0$ (A)

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = \int_{t_1}^{t_2} dt \sum_i \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right\} = 0 \Rightarrow$$

$$\delta S = \int_{t_1}^{t_2} dt \sum_i \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) \right\} = 0 \Rightarrow$$

$$\delta S = \int_{t_1}^{t_2} dt \sum_i \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \cdot (-1) \right\} = 0 \Rightarrow$$

$$\delta S = \sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_{t_1}^{t_2}}_{0\ (A)} + \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right\} \delta q_i = 0$$

$$\therefore \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$ Lagrange Equations

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Examples:

Free particle:

$$L = \frac{1}{2} m \dot{x}^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad (\text{momentum}); \quad \frac{\partial L}{\partial x} = 0$$

$$\therefore \frac{d}{dt}(m\dot{x}) = 0 \Rightarrow m\ddot{x} = 0$$

Harmonic Oscillator:

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 = T - V$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}; \quad \frac{\partial L}{\partial x} = -kx \Rightarrow$$

$$\frac{d}{dt}(m\dot{x}) - (-kx) = 0 \Rightarrow \boxed{m\ddot{x} + kx = 0}$$ H.O. Equation

$$\ddot{x} + \frac{k}{m} x = 0$$

$$\omega^2 = k/m$$

Conservation Laws in Classical Mechanics: Energy and Momentum Conservation

Conservation Laws (3)

Symmetries of the Lagrangian of a system result to conserved quantities ⇒ Conservation Laws

Homogeneity of time: "The Lagrangian of a closed system cannot depend explicitly on time"

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \cancel{\frac{\partial L}{\partial t}} \Rightarrow$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \Rightarrow$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \Rightarrow$$

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = 0$$

$$\frac{dH}{dt} = 0$$

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \quad \text{(Hamiltonian)}$$

Energy is conserved!!!

Homogeneity in Space: The mechanical properties of a closed system are unchanged under: (4)

$$\vec{r} \to \vec{r}\,' = \vec{r} + \vec{\epsilon}$$

$$\delta\vec{r} = \vec{\epsilon}$$

$$\delta L = \sum_a \frac{\partial L}{\partial \vec{r}_a}\delta\vec{r}_a = \vec{\epsilon}\cdot\sum_a \frac{\partial L}{\partial \vec{r}_a}$$

$$\delta L = 0 \Rightarrow \sum_a \frac{\partial L}{\partial \vec{r}_a}\delta\vec{r}_a = \vec{\epsilon}\cdot\sum_a \frac{\partial L}{\partial \vec{r}_a} = 0 \Rightarrow$$

$$\sum_a \frac{\partial L}{\partial \vec{r}_a} = 0 \overset{\text{L.E}}{\Rightarrow} \sum_a \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_a}\right) = 0$$

$$\therefore \frac{d}{dt}\left(\sum_a \frac{\partial L}{\partial \vec{v}_a}\right) = 0$$

Momentum is Conserved !!

Conservation Laws in Classical Mechanics
Angular Momentum Conservation

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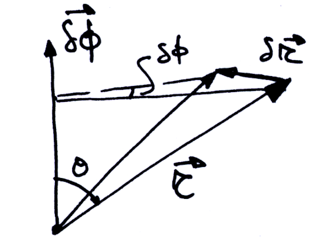
Isotropy in Space: The mechanical properties of a closed system do not vary when it is rotated as a whole.



$$\delta\vec{r} = \vec{\delta\phi}\times\vec{r}$$
$$\delta\vec{V} = \vec{\delta\phi}\times\vec{V}$$

$$\hat{e}\,|\delta\vec{r}| = \delta\phi\cdot r\cdot\sin\theta\,\hat{e} = \vec{\delta\phi}\times\vec{r}$$

$$\delta L = \sum_a\left(\frac{\partial L}{\partial\vec{r}_a}\cdot\delta\vec{r}_a + \frac{\partial L}{\partial\vec{V}_a}\delta\vec{V}_a\right) = 0$$

$$\delta L = \sum_a\left(\dot{\vec{P}}_a\,\delta\vec{r}_a + \vec{P}_a\cdot\delta\vec{V}_a\right)$$

$$\delta L = \sum_a\left(\dot{\vec{P}}_a\cdot\vec{\delta\phi}\times\vec{r}_a + \vec{P}_a\cdot\vec{\delta\phi}\times\vec{V}_a\right)$$

$$\delta L = \sum_a\left(\vec{\delta\phi}\cdot(\vec{r}_a\times\dot{\vec{P}}_a) + \vec{\delta\phi}\cdot(\vec{V}_a\times\vec{P}_a)\right)$$

$$\delta L = \vec{\delta\phi}\cdot\sum_a \vec{r}_a\times\dot{\vec{P}}_a + \dot{\vec{r}}_a\times\vec{P}_a$$

$$\left.\begin{aligned}\delta L &= \vec{\delta\phi}\cdot\frac{d\vec{L}}{dt} = 0\\ \vec{L} &= \sum_a \vec{r}_a\times\vec{P}_a\end{aligned}\right\}\ \text{Angular Momentum is conserved}$$

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Summary:

1. Equations of motion are derived by the Lagrange equations acting on scalar Lagrangians

2. Symmetries of the Lagrangian lead to conservation laws

Symmetries in Quantum Mechanics: Rotations

Examples of Symmetries in Quantum Mechanics ①

Consider a rotation : $\vec{r}' = \vec{r} + \vec{a} \times \vec{r}$

Define the vector operator $\vec{r}_{op}$ such that

$$\vec{r}_{op} |\vec{r}_0\rangle = \vec{r}_0 |\vec{r}_0\rangle$$

$$\vec{r}'_{op} |\vec{r}_0\rangle = (\vec{r}_{op} + \vec{a} \times \vec{r}_{op}) |\vec{r}_0\rangle$$

$$= (\vec{r}_0 + \vec{a} \times \vec{r}_0) |\vec{r}_0\rangle$$

- Therefore $|\vec{r}_0\rangle$ is the eigenstate of both $\vec{r}_{op}$ and $\vec{r}'_{op}$, the rotated operator
- Recall the angular momentum operator

$$L_i = -i\hbar \epsilon_{ijk} X_j \partial_k \quad (\hat{L} = \vec{r} \times \vec{p})$$

$$[L_i, X_j]\psi = L_i(x_j\psi) - X_j L_i \psi$$

$$= -i\hbar \epsilon_{i\ell m} X_\ell \partial_m (x_j \psi) +$$

$$+ i\hbar X_j \epsilon_{i\ell m} X_\ell \partial_m \psi$$

$$= -i\hbar \epsilon_{i\ell m} X_\ell (\delta_{mj}\psi + \cancel{X_j \partial_m \psi})$$

$$+ i\hbar X_j \cancel{\epsilon_{i\ell m} X_\ell \partial_m \psi}$$

$$= -i\hbar \epsilon_{i\ell j} X_\ell = +i\hbar \epsilon_{ij\ell} X_\ell$$

$$\therefore \quad [L_i, X_j] = +i\hbar \epsilon_{ij\ell} X_\ell$$

②

$$\left.\begin{aligned} &[n_i L_i, X_j] = i\hbar\, n_i \epsilon_{ijk} X_k \\ &\hat{n} \text{ is some unit vector} \end{aligned}\right\} \Rightarrow$$

$$[\hat{n}\cdot\vec{L}, X_j] = i\hbar \epsilon_{jki} X_k n_i = i\hbar (\vec{X} \times \vec{n})_j$$

$$\boxed{[\hat{n}\cdot\vec{L}, \vec{X}] = i\hbar (\vec{X} \times \vec{n})}$$

<u>Therefore:</u>

$$\vec{r}'_{op} = \vec{r}_{op} + \vec{a} \times \vec{r}_{op}$$

$$\vec{r}'_{op} = \vec{r}_{op} + \frac{i}{\hbar} [\vec{a}\cdot\vec{L}, \vec{r}]$$

$$\vec{r}'_{op} = e^{i/\hbar\, \vec{a}\cdot\vec{L}}\, \vec{r}_{op}\, e^{-i/\hbar\, \vec{a}\cdot\vec{L}}$$

So in general for any vector operator

$$\boxed{\vec{A}'_{op} = e^{i/\hbar\, \vec{\epsilon}\cdot\vec{G}}\, A_{op}\, e^{-i/\hbar\, \vec{\epsilon}\cdot\vec{G}}}$$

States and Operators under Rotation

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Back to the position operator and angular Momentum

$$\langle \vec{r}_0 | \, e^{+\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}} \, \vec{r}_{op} \, e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}} = \langle \vec{r}_0 | \, \vec{r}_{op}\,' $$

$$= \vec{r}_0\,' \langle \vec{r}_0 | \Rightarrow$$

$$\underbrace{\langle \vec{r}_0 | \, e^{\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}}} \, \vec{r}_{op} = \vec{r}_0\,' \underbrace{\langle \vec{r}_0 | \, e^{i/\hbar\,\vec{\alpha}\cdot\vec{L}}}_{\text{ROTATED EIGENSTATE}}$$

SUMMARY

under rotations:

$$\vec{A}\,' = e^{\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}} \, \vec{A} \, e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}}$$

$$|\psi'\rangle = e^{-i/\hbar\,\vec{\alpha}\cdot\vec{L}} \, |\psi\rangle$$

$\vec{L}$ is the generator of rotations

$$|\psi'\rangle \cong \left(1 - \frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}\right)|\psi\rangle$$

$$\therefore \quad \delta|\psi\rangle = |\psi'\rangle - |\psi\rangle = -\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}\,|\psi\rangle$$

This is why we call it generator

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Consider any transformation acting upon operators and eigenvectors as

$$|\psi'\rangle = U|\psi\rangle \qquad \text{I}$$

$$A' = U^{+} A U \qquad \text{II}$$

$$U = e^{-i\,\varepsilon\cdot G} \qquad \text{III}$$

If this transformation does not change the probabilities then

$$|\langle \phi|\psi\rangle|^2 \overset{?}{=} |\langle \phi'|\psi'\rangle|^2 \Rightarrow$$

$$U^{+}U = \pm 1 \quad \text{(Consider +1, unitary)}$$

$$\therefore \quad G = \text{Hermitean}$$

$$\text{II}\ \text{III} \Rightarrow \quad \delta A = \frac{i\epsilon}{\hbar}[G, A]$$

and $\delta A = 0$ if $[G, A] = 0$

e.g. $\delta H = 0$ if $[G, H] = 0$

Generator of an operation that leaves H invariant (Hamiltonian)

Addition of Angular Momenta I

⑤

The other way around is also true

$$i\hbar \frac{d}{dt}\langle G\rangle = \langle [G,H]\rangle$$

H generates time translations

if $[G,H]=0 \Rightarrow G$ is conserved

ADDITION OF ANGULAR MOMENTA

Suppose we now have a system of two objects each with angular momentum J_1 and J_2 (obviously $[\vec{J}_1, \vec{J}_2]=0$)

$\boxed{\text{Consider for example a system of two fermions each with } S=1/2}$

The possible spin states are

$$|\uparrow\uparrow\rangle \quad |\downarrow\downarrow\rangle \quad |\uparrow\downarrow\rangle \quad |\downarrow\uparrow\rangle$$

which of course are linearly independent.

Another way to represent these states is in the $|j_1 j_2 m_1 m_2\rangle$ notation

⑥

Hence,

$$\left.\begin{array}{l} |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\rangle \\ |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}\rangle \\ |\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}\rangle \\ |\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}\rangle \end{array}\right\} \text{Ⓐ}$$

$|j_1 j_2 m_1 m_2\rangle$

We want to go from these states to the $|J_1 J_2 J M\rangle$ or simply $|JM\rangle$ states of total angular momentum.

Why? Because in many cases J, M are conserved quantum numbers because the Hamiltonian commutes with J^2, J_z

So what is $|JM\rangle$ in terms of Ⓐ?

$$\left.\begin{array}{l} S_z|\uparrow\uparrow\rangle = (\tfrac{1}{2}+\tfrac{1}{2}) = |\uparrow\uparrow\rangle \\ S_z|\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle \\ S_z|\uparrow\downarrow\rangle = S_z|\downarrow\uparrow\rangle = 0 \end{array}\right\} \text{Ⓐ are eigenstates of } S_z$$

What about $\vec{S}^2$?

Addition of Angular Momenta II

(7)

Try $\vec{S}^2|\uparrow\uparrow\rangle = (\vec{S}_1+\vec{S}_2)^2|\uparrow\uparrow\rangle$

$= (\vec{S}_1^2+\vec{S}_2^2+2\vec{S}_1\vec{S}_2)|\uparrow\uparrow\rangle$

$= \left(\frac{1}{2}(\frac{1}{2}+1)+\frac{1}{2}(\frac{1}{2}+1)+2\vec{S}_1\vec{S}_2\right)|\uparrow\uparrow\rangle$

$= \left(\frac{3}{2}+2S_{1z}S_{2z}+2S_{1x}S_{2x}+2S_{1y}S_{2y}\right)|\uparrow\uparrow\rangle$ (a)

$S_{1+} = S_{1x}+iS_{1y}$ $\quad S_{2+} = S_{2x}+iS_{2y}$

$S_{1-} = S_{1x}-iS_{1y}$ $\quad S_{2-} = S_{2x}-iS_{2y}$

$$\begin{aligned} & S_{1+}S_{2-} = S_{1x}S_{2x}+S_{1y}S_{2y}-i\cancel{S_{1x}S_{2y}}+i\cancel{S_{1y}S_{2x}} \\ + \; & S_{1-}S_{2+} = S_{1x}S_{2x}+S_{1y}S_{2y}+i\cancel{S_{1x}S_{2y}}-i\cancel{S_{2x}S_{1y}} \\ \hline & 2(S_{1x}S_{2x}+S_{1y}S_{2y}) = S_{1+}S_{2-}+S_{1-}S_{2+} \end{aligned}$$

(β)

(a) (β) $\Rightarrow \vec{S}^2|\uparrow\uparrow\rangle = \left[\frac{3}{2}+2\underbrace{S_{1z}}_{\frac{1}{2}}\underbrace{S_{2z}}_{\frac{1}{2}}+\underbrace{S_{1+}S_{2-}}_{\to 0}+\underbrace{S_{1-}S_{2+}}_{\to 0}\right]|\uparrow\uparrow\rangle$

$$\boxed{\begin{array}{l} \vec{S}^2|\uparrow\uparrow\rangle = 1(1+1)|\uparrow\uparrow\rangle \\ \text{which means that} \\ \quad |1+1\rangle = |\uparrow\uparrow\rangle \end{array}}$$

(8)

IN CONCLUSION:

$$|1\,1\rangle = |\uparrow\uparrow\rangle$$

and also in the same way $|1-1\rangle = |\downarrow\downarrow\rangle$

we still need two more states to find.

Recall

$S_-|SM\rangle = \sqrt{S(S+1)-M(M-1)}\,|S,M-1\rangle$

$$\left.\begin{array}{l} S_-|\uparrow\uparrow\rangle = S_-|11\rangle = \sqrt{1(1+1)-1(1-1)}\,|1\,0\rangle \\ \therefore |1\,0\rangle = \frac{1}{\sqrt{2}}S_-|11\rangle \end{array}\right\} \Rightarrow$$

$S_-|11\rangle = (S_{1-}+S_{2-})|\uparrow\uparrow\rangle$

$|10\rangle = \frac{1}{\sqrt{2}}(S_{1-}+S_{2-})|\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}}\sqrt{\frac{1}{2}(\frac{1}{2}+1)-\frac{1}{2}(\frac{1}{2}-1)}\,\times$

$\times\,(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)$$

Addition of Angular Momenta III

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Therefore for total spin $S=1$ we have the following states arranged in a TRIPLET:

$$|1\,1\rangle = |\uparrow\uparrow\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|1\,{-1}\rangle = |\downarrow\downarrow\rangle$$

The 4^{th} state is a singlet and can be derived by requiring that it must be orthogonal to the others

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

All these mean that if you add two $S=1/2$ angular momenta the result $= 0, 1$ or if you prefer group theory language $2\otimes 2 = 1\oplus 3$

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In the same way if you combine three $s=\frac{1}{2}$ particles you get

$$\begin{aligned} 2\otimes 2\otimes 2 &= (3\oplus 1)\otimes 2 \\ &= (3\otimes 2)\oplus(1\otimes 2) \\ &= (4\oplus 2)\oplus 2 \\ &= \underset{\substack{\downarrow \\ \text{spin } \frac{3}{2} \\ \text{quartet}}}{4} \oplus \underbrace{2\oplus 2}_{\substack{\text{spin } \frac{1}{2} \\ \text{doublets}}} \end{aligned}$$

CLEBSCH-GORDAN COEFFICIENTS

IN GENERAL:

$$|J_1 J_2 m_1 m_2\rangle = \sum_{\substack{J \\ m=m_1+m_2}} |J_1 J_2 J M\rangle \underbrace{\langle J_1 J_2 J M | J_1 J_2 m_1 m_2\rangle}_{\substack{\text{CLEBSCH-} \\ \text{GORDAN} \\ \text{COEFFICIENTS}}}$$

There are $(2J_1+1)(2J_2+1)$ diff. $|JM\rangle$ states with

$$|J_1 - J_2| \le J \le |J_1 + J_2|$$

$$m = m_1 + m_2$$

Isotopic Spin in Strong Interactions I

ISOTOPIC SPIN IN STRONG INTERACTIONS (11)

1. $m_p \approx m_n$
2. Protons and Neutrons behave the same under strong interactions (SI)
3. The interaction does not depend on charge (Charge is different but SI dominates

$\therefore$ p n are one and the same particle the Nucleon but

$$p \to I_3 = 1/2$$
$$n \to I_3 = -1/2$$

$$I_3^{(N)}|p\rangle = \tfrac{1}{2}|p\rangle \quad I_3^{(N)}|n\rangle = -\tfrac{1}{2}|n\rangle$$

And the angular momentum algebra holds

$$[I_i^{(N)} I_j^{(N)}] = i\,\epsilon_{ijk} I_k^{(N)}$$

$$(I^{(N)})^2|p\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)|p\rangle$$
$$(I^{(N)})^2|n\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)|n\rangle$$

$$Q_N = I_3^{(N)} + \tfrac{1}{2}$$

Similar things for $\pi^\pm \pi^0$ (12)

	I_3
π^+	1
π^0	0
π^-	−1

$$I_3^{(\pi)}|\pi^+\rangle = |\pi^+\rangle$$
$$I_3^{(\pi)}|\pi^0\rangle = 0$$
$$I_3^{\pi}|\pi^-\rangle = -|\pi^-\rangle$$
$$I_\pm^{(\pi)}|\pi^0\rangle = \sqrt{2}\,|\pi^\pm\rangle$$

$I^{(\pi)} = 1 \quad I^{(N)} = \frac{1}{2} \quad \therefore \quad 3 \otimes 2 = 4 \oplus 2$

6-states

$$\left.\begin{aligned}
|\tfrac{3}{2}\tfrac{3}{2}\rangle &= |\pi^+ p\rangle \\
|\tfrac{3}{2}\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}}\,|\pi^+ n\rangle + \sqrt{\tfrac{2}{3}}\,|\pi^0 p\rangle \\
|\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle &= \sqrt{\tfrac{1}{3}}\,|\pi^- p\rangle + \sqrt{\tfrac{2}{3}}\,|\pi^0 n\rangle \\
|\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle &= |\pi^- n\rangle
\end{aligned}\right\} 4$$

$$\left.\begin{aligned}
|\tfrac{1}{2}\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}\,|\pi^+ n\rangle - \sqrt{\tfrac{1}{3}}\,|\pi^0 p\rangle \\
|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle &= \sqrt{\tfrac{1}{3}}\,|\pi^0 n\rangle - \sqrt{\tfrac{2}{3}}\,|\pi^- p\rangle
\end{aligned}\right\} 2$$

Isotopic Spin II

(13)

And the inverse:

$$|\pi^+p\rangle = |\tfrac{3}{2}\tfrac{3}{2}\rangle$$

$$|\pi^0p\rangle = \sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$|\pi^-p\rangle = \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle$$

$$|\pi^+n\rangle = \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle = |\pi^+n\rangle$$

$$|\pi^0n\rangle = \sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle$$

$$|\pi^-n\rangle = |\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle$$

Experimentaly we know that the strong interaction cannot change I, I_3 and does not depend upon I_3 (Charge)

In other words the theory is independent upon rotations in the Isospin-space that is

$$\langle I', I_3' |\, H_{INT} \,| I I_3\rangle = a_I\, \delta_{II'}\, \delta_{I_3 I_3'}$$

(14)

Lets see where can we use these stuff:

Consider the reactions

$$\left.\begin{array}{l} \pi^+p \to \pi^+p \\ \pi^-p \to \pi^-p \\ \pi^-p \to n\pi^0 \end{array}\right\}$$

$|j_1 j_2 m_1 m_2\rangle$ States which need to be written in terms of $|J M\rangle$ which are the Cons. Quantum numbers

$$A_{\pi^+p\to\pi^+p} \sim \langle \pi^+p|\pi^+p\rangle \sim \langle\tfrac{3}{2}\tfrac{3}{2}|\tfrac{3}{2}\tfrac{3}{2}\rangle \sim a_{3/2}$$

$$\therefore\ \sigma(\pi^+p\to\pi^+p) \sim |a_{3/2}|^2 \quad \text{(I)}$$

$$A_{\pi^-p\to\pi^-p} \sim \Big(\sqrt{\tfrac{1}{3}}\langle\tfrac{3}{2}\,{-\tfrac{1}{2}}|\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle\sqrt{\tfrac{1}{3}}\, a_{3/2} + \sqrt{\tfrac{2}{3}}\langle\tfrac{1}{2}\,{-\tfrac{1}{2}}|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\sqrt{\tfrac{2}{3}}\, a_{1/2}\Big)$$

$$A_{\pi^-p\to\pi^-p} \sim \tfrac{1}{3}a_{3/2} + \tfrac{2}{3}a_{1/2}$$

$$\sigma(\pi^-p\to\pi^-p) \sim \tfrac{1}{9}|a_{3/2} + 2a_{1/2}|^2 \quad \text{(II)}$$

Isotopic Spin III

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$$A_{\pi^- p \to \pi^0 n} = \left[\sqrt{\tfrac{1}{3}} \sqrt{\tfrac{2}{3}} \left\langle \tfrac{3}{2} - \tfrac{1}{2} \middle| \tfrac{3}{2} - \tfrac{1}{2} \right\rangle a_{3/2} + \right.$$
$$\left. - \sqrt{\tfrac{1}{3}} \sqrt{\tfrac{2}{3}}\, a_{1/2} \right]$$

$$A_{\pi^- p \to \pi^0 n} = \frac{\sqrt{2}}{3} a_{3/2} - \frac{\sqrt{2}}{3} a_{1/2}$$

$$\therefore\ \sigma(\pi^- p \to \pi^0 n) = \frac{2}{9} \left| a_{3/2} - a_{1/2} \right|^2 \quad \text{III}$$

if $a_{1/2} \ll a_{3/2} \Rightarrow$

$$\sigma(\pi^+ p \to \pi^+ p) \sim \left| a_{3/2} \right|^2$$
$$\sigma(\pi^- p \to \pi^- p) \sim \frac{1}{9} \left| a_{3/2} \right|^2$$
$$\sigma(\pi^- p \to \pi^0 n) \sim \frac{2}{9} \left| a_{3/2} \right|^2$$

$$\boxed{\sigma(\pi^+ p \to \pi^+ p) : \sigma(\pi^- p \to \pi^- p) : \sigma(\pi^- p \to \pi^0 n) = 9 : 1 : 2}$$

ISOSPIN SYMMETRY $\rightarrow$ ISOSPIN QUANTUM NUMBERS $\rightarrow$ Cross-section ratios

Isotopic Spin IV: Experimental Evidence

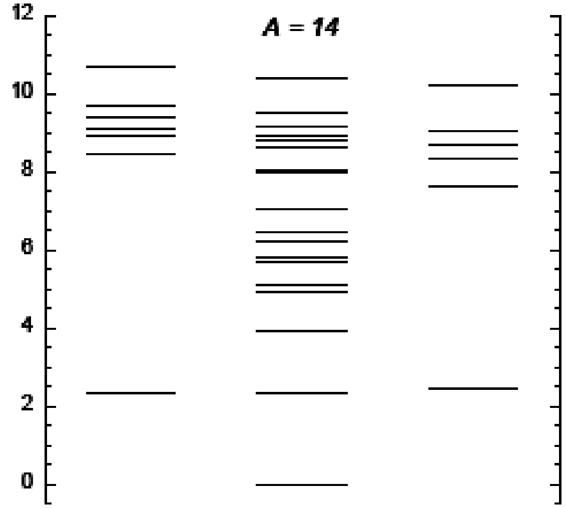

Figure 10. Simplified isobar diagram for the $A = 14$ nuclei. The presumed $I = 1$ isobaric analogue levels are shown in grey. Following the usual practice, the diagrams for individual isobars are shifted vertically to eliminate the n-p mass difference and the Coulomb energy.

$$^{14}\mathrm{O} : {}^{12}\mathrm{C} + (pp) \;\; I_3 = +1$$
$$^{14}\mathrm{N} : {}^{12}\mathrm{C} + (pn) \;\; I_3 = 0$$
$$^{14}\mathrm{C} : {}^{12}\mathrm{C} + (nn) \;\; I_3 = -1$$

Taken from C. Quigg's paper: hep-ex/0204104