Problem set III

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Problem 1: We have already shown in class that the gauge transformation for a Yang-Mills Field is:

$$A'_{\mu} = U(\theta) \cdot A_{\mu} \cdot U^{-1}(\theta) - \left(\frac{i}{g}\right) \partial_{\mu} U(\theta) \cdot U^{-1}(\theta)$$

This transformation leaves the scalar Lagrangian invariant under:
$\Phi \rightarrow \Phi' = U(\theta) \cdot \Phi$ where $U(\theta) = \mathrm{e}^{-i \vec{\theta} \cdot \vec{\tau}}$ and $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the **SU(2)** generators. Show that the gauge field Lagrangian given by: $L = -\frac{1}{4} Tr(F^{\mu\nu} F_{\mu\nu})$ is also invariant under this transformation, where:

$$F_{\mu\nu} = \tau^i F^i_{\mu\nu} \text{ and } F^i_{\mu\nu} = \partial_{\mu} A^i_{\nu} - \partial_{\nu} A^i_{\mu} + g \epsilon^{ijk} A^j_{\mu} A^k_{\nu}$$

Hints:

(a) Show that the Y-M field change under U is given by

$$\delta A^i_{\mu} = \epsilon^{ijk} \theta^j A^k_{\mu} - \frac{1}{g} \partial_{\mu} \theta^i$$

(b) Calculate:

$$\delta(\partial_{\mu} A^i_{\nu} - \partial_{\nu} A^i_{\mu}) = \epsilon^{ijk} \theta^j (\partial_{\mu} A^k_{\nu} - \partial_{\nu} A^k_{\mu}) + \epsilon^{ijk} [(\partial_{\mu} \theta^j) \cdot A^k_{\nu} - (\partial_{\nu} \theta^j) \cdot A^k_{\mu}]$$

(c) Calculate:

$$\epsilon^{ijk} \delta(A^j_{\mu} A^k_{\nu}) = -\frac{\epsilon^{ijk}}{g} [(\partial_{\mu} \theta^j) A^k_{\nu} - (\partial_{\nu} \theta^j) A^k_{\mu}] +$$

$$+ \epsilon^{ijk} \epsilon^{jlm} \theta^l A^k_{\nu} A^m_{\mu} + \epsilon^{ijk} \epsilon^{klm} A^m_{\nu} A^j_{\mu} \theta^l$$

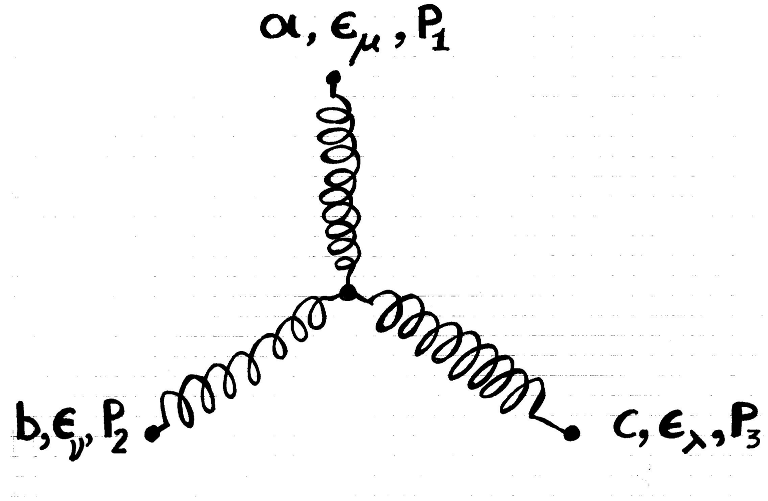
(d) Use the **SU(2)** group generator relations:

$$[\tau^i, \tau^j] = i \epsilon^{ijk} \tau^k \,;\; (\tau^i)_{jk} = -i \epsilon_{ijk} \,;\, Tr(\tau^i \tau^j) = k \delta^{ij}$$

(e) Calculate the change of the Lagrangian.

$$\delta F^i_{\mu\nu} = \epsilon^{ijk} \theta^j F^k_{\mu\nu}$$

Problem 2: Show that a non-abelian theory based on an SU(N) symmetry group predicts diagrams of the type:



which corresponds to the following Feynman rule:

$$-ig\, f_{abc}[g_{\mu\nu}(p_1-p_2)_\lambda+g_{\nu\lambda}(p_2-p_3)_\mu+g_{\lambda\mu}(p_3-p_1)_\nu]$$