Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 6 Answers

1. The free Lagrangian density for the two massless fermions is

$$\begin{aligned}\mathcal{L} &= i\overline{\psi}_\nu\gamma^\mu(\partial_\mu\psi_\nu) + i\overline{\psi}_e\gamma^\mu(\partial_\mu\psi_e)\\ &= i\begin{pmatrix}\overline{\psi}_\nu & \overline{\psi}_e\end{pmatrix}\gamma^\mu\partial_\mu\begin{pmatrix}\psi_\nu\\ \psi_e\end{pmatrix}\\ &= i\overline{\Psi}\gamma^\mu(\partial_\mu\Psi)\end{aligned}$$

For a special unitary transformation

$$\Psi' = U\Psi$$

where

$$U = e^{-i\alpha_i\sigma_i/2}$$

then, since U is unitary, meaning $U^\dagger U = I$, and $U \neq U(x^\mu)$

$$\begin{aligned}\mathcal{L}' &= i\overline{\Psi}'\gamma^\mu(\partial_\mu\Psi')\\ &= i\overline{\Psi}U^\dagger\gamma^\mu(\partial_\mu U\Psi)\\ &= i\overline{\Psi}\gamma^\mu(\partial_\mu\Psi) = \mathcal{L}\end{aligned}$$

Hence, the Lagrangian density is invariant to tranformations under U. Using

$$\frac{\partial\Psi}{\partial\alpha_i} = -\frac{i}{2}\sigma_i\Psi, \qquad \frac{\partial\overline{\Psi}}{\partial\alpha_i} = \frac{i}{2}\overline{\Psi}\sigma_i$$

and

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = i\overline{\Psi}\gamma^\mu, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Psi})} = 0$$

then the Nöther currents associated with these transformations are

$$J^\mu_{Vi} = (i\overline{\Psi}\gamma^\mu)(-\frac{i}{2}\sigma_i\Psi) + (\frac{i}{2}\overline{\Psi}\sigma_i)(0) = \frac{1}{2}\overline{\Psi}\gamma^\mu\sigma_i\Psi$$

In terms of ψ_ν and ψ_e, these are therefore

$$\begin{aligned}J^\mu_{V1} &= \frac{1}{2}\left(\overline{\psi}_\nu\gamma^\mu\psi_e + \overline{\psi}_e\gamma^\mu\psi_\nu\right)\\ J^\mu_{V2} &= \frac{i}{2}\left(\overline{\psi}_e\gamma^\mu\psi_\nu - \overline{\psi}_\nu\gamma^\mu\psi_e\right)\\ J^\mu_{V3} &= \frac{1}{2}\left(\overline{\psi}_\nu\gamma^\mu\psi_\nu - \overline{\psi}_e\gamma^\mu\psi_e\right)\end{aligned}$$

The corresponding axial currents are

$$\begin{aligned}J^\mu_{A1} &= \frac{1}{2}\left(\overline{\psi}_\nu\gamma^\mu\gamma^5\psi_e + \overline{\psi}_e\gamma^\mu\gamma^5\psi_\nu\right)\\ J^\mu_{A2} &= \frac{i}{2}\left(\overline{\psi}_e\gamma^\mu\gamma^5\psi_\nu - \overline{\psi}_\nu\gamma^\mu\gamma^5\psi_e\right)\\ J^\mu_{A3} &= \frac{1}{2}\left(\overline{\psi}_\nu\gamma^\mu\gamma^5\psi_\nu - \overline{\psi}_e\gamma^\mu\gamma^5\psi_e\right)\end{aligned}$$

and hence we can form the combinations

$$J_i^\mu = \frac{1}{2}\left(J_{Vi}^\mu - J_{Ai}^\mu\right)$$

which are also conserved. Under local gauge invariance, these currents can couple to fields W_i^μ. The charged interaction term for the currents J_1^μ and J_2^μ is

$$\begin{aligned}
\mathcal{L}_C &= g_W W_{1\mu} J_1^\mu + g_W W_{2\mu} J_2^\mu \\
&= \frac{g_W}{2} W_{1\mu}\left[\overline{\psi}_\nu \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_e + \overline{\psi}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_\nu\right] + \\
&\quad \frac{g_W}{2} W_{2\mu} i\left[\overline{\psi}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_\nu - \overline{\psi}_\nu \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_e\right] \\
&= \frac{g_W}{2}\left[\left(W_{1\mu} - iW_{2\mu}\right)\overline{\psi}_\nu \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_e + \left(W_{1\mu} + iW_{2\mu}\right)\overline{\psi}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_\nu\right] \\
&= \frac{g_W}{2}\left(\sqrt{2}W_\mu^- \overline{\psi}_\nu \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_e + \sqrt{2}W_\mu^+ \overline{\psi}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_\nu\right) \\
&= \frac{g_W}{\sqrt{2}}\left[W_\mu^- \overline{\psi}_\nu \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_e + W_\mu^+ \overline{\psi}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\psi_\nu\right] \\
&= \frac{g_W}{\sqrt{2}}\left[W_\mu^- \overline{\psi}_{\nu L} \gamma^\mu \psi_{eL} + W_\mu^+ \overline{\psi}_{eL} \gamma^\mu \psi_{\nu L}\right]
\end{aligned}$$

as is observed.

The current which interacts with the Y^μ field is generally

$$J_Y^\mu = a\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} + b\overline{\psi}_{\nu R}\gamma^\mu\psi_{\nu R} + c\overline{\psi}_{eL}\gamma^\mu\psi_{eL} + d\overline{\psi}_{eR}\gamma^\mu\psi_{eR}$$

The total neutral interaction term is then

$$\mathcal{L}_N = g_W W_{3\mu} J_3^\mu + \frac{g_Y}{2} Y_\mu J_Y^\mu$$

In terms of the left- and right-handed terms, this is

$$\begin{aligned}
\mathcal{L}_N &= \frac{g_W}{2} W_{3\mu}\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} - \frac{g_W}{2} W_{3\mu}\overline{\psi}_{eL}\gamma^\mu\psi_{eL} + a\frac{g_Y}{2} Y_\mu \overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} \\
&+ b\frac{g_Y}{2} Y_\mu \overline{\psi}_{\nu R}\gamma^\mu\psi_{\nu R} + c\frac{g_Y}{2} Y_\mu \overline{\psi}_{eL}\gamma^\mu\psi_{eL} + d\frac{g_Y}{2} Y_\mu \overline{\psi}_{eR}\gamma^\mu\psi_{eR}
\end{aligned}$$

Rotating the W_3^μ and Y^μ fields into each other, then

$$Z^\mu = W_3^\mu \cos\theta_W - Y^\mu \sin\theta_W, \qquad A^\mu = W_3^\mu \sin\theta_W + Y^\mu \cos\theta_W$$

which can be inverted to give

$$W_3^\mu = Z^\mu \cos\theta_W + A^\mu \sin\theta_W, \qquad Y^\mu = -Z^\mu \sin\theta_W + A^\mu \cos\theta_W$$

Therefore

$$\begin{aligned}
\mathcal{L}_N &= \frac{g_W}{2}\left(Z_\mu\cos\theta_W + A_\mu\sin\theta_W\right)\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} - \frac{g_W}{2}\left(Z_\mu\cos\theta_W + A_\mu\sin\theta_W\right)\overline{\psi}_{eL}\gamma^\mu\psi_{eL} \\
&+ a\frac{g_Y}{2}\left(-Z_\mu\sin\theta_W + A_\mu\cos\theta_W\right)\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} + b\frac{g_Y}{2}\left(-Z_\mu\sin\theta_W + A_\mu\cos\theta_W\right)\overline{\psi}_{\nu R}\gamma^\mu\psi_{\nu R} \\
&+ c\frac{g_Y}{2}\left(-Z_\mu\sin\theta_W + A_\mu\cos\theta_W\right)\overline{\psi}_{eL}\gamma^\mu\psi_{eL} + d\frac{g_Y}{2}\left(-Z_\mu\sin\theta_W + A_\mu\cos\theta_W\right)\overline{\psi}_{eR}\gamma^\mu\psi_{eR}
\end{aligned}$$

which can be expressed as

$$\begin{aligned}\mathcal{L}_N &= A_\mu \frac{1}{2}\left(g_W \sin\theta_W + a g_Y \cos\theta_W\right)\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} + bA_\mu\frac{1}{2}g_Y\cos\theta_W\overline{\psi}_{\nu R}\gamma^\mu\psi_{\nu R} \\ &+ A_\mu \frac{1}{2}\left(-g_W \sin\theta_W + c g_Y \cos\theta_W\right)\overline{\psi}_{eL}\gamma^\mu\psi_{eL} + dA_\mu\frac{1}{2}g_Y\cos\theta_W\overline{\psi}_{eR}\gamma^\mu\psi_{eR} \\ &+ Z_\mu \frac{1}{2}\left(g_W \cos\theta_W - a g_Y \sin\theta_W\right)\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} - bZ_\mu\frac{1}{2}g_Y\sin\theta_W\overline{\psi}_{\nu R}\gamma^\mu\psi_{\nu R} \\ &+ Z_\mu \frac{1}{2}\left(-g_W \cos\theta_W - c g_Y \sin\theta_W\right)\overline{\psi}_{eL}\gamma^\mu\psi_{eL} - dZ_\mu\frac{1}{2}g_Y\sin\theta_W\overline{\psi}_{eR}\gamma^\mu\psi_{eR}\end{aligned}$$

The neutrino couplings to the photon have to be zero for both the left- and right-handed parts, so

$$g_W \sin\theta_W + a g_Y \cos\theta_W = 0, \qquad b = 0$$

The electron terms must be equal to the QED coupling

$$-eA_\mu\overline{\psi}_e\gamma^\mu\psi_e = -eA_\mu\overline{\psi}_e\gamma^\mu\frac{1}{2}(1-\gamma^5+1+\gamma^5)\psi_e = -eA_\mu\left(\overline{\psi}_{eL}\gamma^\mu\psi_{eL} + \overline{\psi}_{eR}\gamma^\mu\psi_{eR}\right)$$

so

$$\frac{1}{2}\left(-g_W\sin\theta_W + c g_Y\cos\theta_W\right) = -e, \qquad \frac{1}{2}d g_Y\cos\theta_W = -e$$

All the above are satisfied by

$$g_W\sin\theta_W = g_Y\cos\theta_W, \qquad e = g_W\sin\theta_W$$

and

$$a = -1, \qquad b = 0, \qquad c = -1, \qquad d = -2$$

as can be verified by direct substitution.

The general term for the Z coupling to neutrinos is

$$\frac{g_Z}{2}Z_\mu\left[c_{\nu L}\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} + c_{\nu R}\overline{\psi}_{\nu R}\gamma^\mu\psi_{\nu R}\right]$$

Since $b = 0$, the Z coupling to the neutrino is purely left-handed and the term is

$$\begin{aligned}\frac{g_Z}{2}Z_\mu c_L\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} &= Z_\mu\left(g_W\cos\theta_W + g_Y\sin\theta_W\right)\frac{1}{2}\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} \\ &= Z_\mu\left(g_W\cos\theta_W + \frac{g_W\sin\theta_W}{\cos\theta_W}\sin\theta_W\right)\frac{1}{2}\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} \\ &= Z_\mu\left(\frac{g_W}{\cos\theta_W}\right)\left(\cos^2\theta_W + \sin^2\theta_W\right)\frac{1}{2}\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} \\ &= \frac{1}{2}\left(\frac{g_W}{\cos\theta_W}\right)Z_\mu\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L}\end{aligned}$$

and so has a strength $g_Z = g_W/\cos\theta_W$ and couplings

$$c_{\nu L} = 1, \qquad c_{\nu R} = 0$$

Using $c_V = (c_L + c_R)/2$ and $c_A = (c_L - c_R)/2$, these are

$$c_{\nu V} = \frac{1}{2}, \qquad c_{\nu A} = \frac{1}{2}$$

The Z coupling terms to the electron are both left- and right-handed

$$\begin{aligned}
& Z_\mu \left(-g_W \cos\theta_W + g_Y \sin\theta_W\right) \frac{1}{2} \overline{\psi}_{eL} \gamma^\mu \psi_{eL} + Z_\mu g_Y \sin\theta_W \overline{\psi}_{eR} \gamma^\mu \psi_{eR} \\
= \; & Z_\mu \left(-g_W \cos\theta_W + \frac{g_W \sin\theta_W}{\cos\theta_W} \sin\theta_W\right) \frac{1}{2} \overline{\psi}_{eL} \gamma^\mu \psi_{eL} + Z_\mu \frac{g_W \sin\theta_W}{\cos\theta_W} \sin\theta_W \overline{\psi}_{eR} \gamma^\mu \psi_{eR} \\
= \; & \frac{1}{2} \left(\frac{g_W}{\cos\theta_W}\right) Z_\mu \left[\left(-\cos^2\theta_W + \sin^2\theta_W\right) \overline{\psi}_{eL} \gamma^\mu \psi_{eL} + 2\sin^2\theta_W \overline{\psi}_{eR} \gamma^\mu \psi_{eR}\right] \\
= \; & \frac{1}{2} \left(\frac{g_W}{\cos\theta_W}\right) Z_\mu \left[\left(-1 + 2\sin^2\theta_W\right) \overline{\psi}_{eL} \gamma^\mu \psi_{eL} + 2\sin^2\theta_W \overline{\psi}_{eR} \gamma^\mu \psi_{eR}\right]
\end{aligned}$$

Hence, the overall strength g_Z is the same and the couplings are

$$c_{eL} = -1 + 2\sin^2\theta_W, \qquad c_{eR} = 2\sin^2\theta_W$$

so that

$$c_{eV} = -\frac{1}{2} + 2\sin^2\theta_W, \qquad c_{eA} = -\frac{1}{2}$$

2. The general formula for the partial width is

$$\Gamma = \frac{|M|^2 \rho}{2M_Z}$$

with the phase space for the two-body decay, which is independent of solid angle, being

$$\rho = \frac{1}{8\pi}$$

Since the matrix element for the Z to decay to $f\overline{f}$ is given as

$$\langle |M|^2 \rangle = \frac{g_W^2 M_Z^2}{3\cos^2\theta_W} \left(c_{fV}^2 + c_{fA}^2\right)$$

the partial width is

$$\Gamma(Z \to f\overline{f}) = \frac{g_W^2 M_Z^2}{3\cos^2\theta_W} \left(c_{fV}^2 + c_{fA}^2\right) \frac{1}{8\pi} \frac{1}{2M_Z} = \frac{g_W^2 M_Z}{48\pi \cos^2\theta_W} \left(c_{fV}^2 + c_{fA}^2\right)$$

Using

$$\cos\theta_W = \frac{M_W}{M_Z}$$

then the partial width becomes

$$\Gamma(Z \to f\overline{f}) = \frac{g_W^2 M_Z^3}{48\pi M_W^2} \left(c_{fV}^2 + c_{fA}^2\right) = \frac{g_W^2}{8M_W^2} \frac{M_Z^3}{6\pi} \left(c_{fV}^2 + c_{fA}^2\right) = \frac{G_F}{\sqrt{2}} \frac{M_Z^3}{6\pi} \left(c_{fV}^2 + c_{fA}^2\right)$$

For any neutrino, $c_{\nu V} = 0.5$ and $c_{\nu A} = 0.5$ so the partial width is

$$\Gamma(Z \to \nu_e \overline{\nu}_e) = 0.166 \text{ GeV}$$

This is the same for the muon and tau neutrinos also.

The mass of the Z is less than twice the top mass so the allowed decays of the Z are to all the charged leptons and neutrinos, $Z \to l^+ l^-$ and $Z \to \nu_l \overline{\nu}_l$, and all the quarks except the top, $Z \to q\overline{q}$. The charged leptons have $c_{eV} = -0.037$ and $c_{eA} = -0.5$ so the partial width is

$$\Gamma(Z \to e^+ e^-) = 0.083 \text{ GeV}$$

and again is the same for the muon and tau also.

The u and c quarks have $c_{uV} = 0.192$ and $c_{uA} = 0.5$ so for one colour

$$\Gamma(Z \to u\overline{u}) = 0.095 \text{ GeV}$$

For d, s and b, $c_{dV} = -0.346$ and $c_{dA} = -0.5$, so

$$\Gamma(Z \to d\overline{d}) = 0.123 \text{ GeV}$$

The Z width to hadrons for all three colours is therefore

$$\Gamma(Z \to \text{hadrons}) = 6\Gamma(Z \to u\overline{u}) + 9\Gamma(Z \to d\overline{d}) = 1.677 \text{ GeV}$$

The visible partial width is thus

$$\Gamma_{visible} = 3\Gamma(Z \to e^+e^-) + \Gamma(Z \to \text{hadrons}) = 1.926 \text{ GeV}$$

The invisible width can be determined from the measured visible and total widths i.e.

$$\Gamma_{invisible} = \Gamma_Z - \Gamma_{visible}$$

Above technique was used to constrain the number of light neutrinos and hence the number of generations.

The total width is

$$\Gamma_Z = 3\Gamma(Z \to \nu_e\overline{\nu}_e) + 3\Gamma(Z \to e^+e^-) + \Gamma(Z \to \text{hadrons}) = 2.424 \text{ GeV}$$

which corresponds to a lifetime of 2.7×10^{-25} s. The leptonic branching fractions are each

$$\mathcal{B}(Z \to \nu_e\overline{\nu}_e) = \frac{\Gamma(Z \to \nu_e\overline{\nu}_e)}{\Gamma_Z} = 6.8\%, \qquad \mathcal{B}(Z \to e^+e^-) = \frac{\Gamma(Z \to e^+e^-)}{\Gamma_Z} = 3.4\%$$

and the branching fraction to hadrons is

$$\mathcal{B}(Z \to \text{hadrons}) = \frac{\Gamma(Z \to \text{hadrons})}{\Gamma_Z} = 69.1\%$$

The fractions of hadronic events containing c and b quarks are

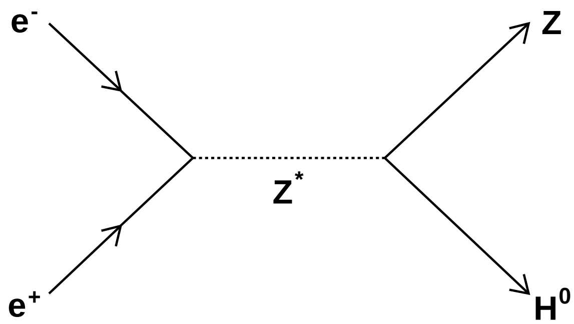
$$R_c = \frac{3\Gamma(Z \to c\overline{c})}{\Gamma(Z \to \text{hadrons})} = 17.0\%, \qquad R_b = \frac{3\Gamma(Z \to b\overline{b})}{\Gamma(Z \to \text{hadrons})} = 22.0\%,$$

The total width and branching fraction to hadrons are very similar to those for $W^\pm$ decay as might be expected from SU(2) symmetry, although the detailed quark production rates, particularly for b quarks, are very different.

3. (i) The coupling of the Higgs to a fermion is proportional to the fermion mass. Hence, the amplitude for a process involving this will go as the mass, while the rate goes as the square of the mass.

   (ii) Principal production mode for the Higgs at LEP2 is $e^+e^- \to Z^{0*} \to Z^0H$ and so Feynman diagram is as overleaf.

   For a given centre of mass energy, E_{cm}, then the Higgs mass must be $m_H < E_{cm} - M_Z$, which for $E = 209$ GeV means $m_H < 118$ GeV.



From energy and momentum conservation in the centre of mass

$$E_{cm} = E_H + E_Z \qquad p_H = p_Z$$

Squaring the second of these

$$E_H^2 - m_H^2 = E_Z^2 - m_Z^2$$

so

$$m_H^2 - m_Z^2 = E_H^2 - E_Z^2 = (E_H + E_Z)(E_H - E_Z) = E_{cm}(E_H - E_Z)$$

Hence

$$E_H - E_Z = \frac{m_H^2 - m_Z^2}{E_{cm}}$$

Adding this to the energy conservation equation gives

$$2E_H = E_{cm} + \frac{m_H^2 - m_Z^2}{E_{cm}}$$

so

$$E_H = \frac{E_{cm}^2 + m_H^2 - m_Z^2}{2E_{cm}}$$

For $m_H = 115$ GeV, $E_H = 116.3$ GeV.

At the kinematic limit, the Higgs and Z^0 would be produced at rest and so have zero phase space and so zero cross-section. Even at 115 GeV, the energy is only slightly greater than the mass and so the phase space is small. The highest limit possible at LEP2 would be somewhat below the kinematic value, at about 115GeV.

(iii) A Higgs of 115 GeV can decay into any quark-antiquark pair, except for $t\bar{t}$, or any lepton-antilepton pair. It is too light to decay to Z^0Z^0 or W^+W^-. Hence, since the rate to any pair is proportional to m^2 and neglecting any differences in the phase space since $m_H \gg 2m_f$, then the branching ratio to a fermion pair $f\overline{f}$ is

$$\mathcal{B}(f\overline{f}) = \frac{m_f^2}{\sum_f m_f^2}$$

where each of the quarks must be included three times in the sum to account for colour. The main decay mode is clearly the one to the heaviest particle, which is the b quark, so the dominant decay is $H \to b\bar{b}$. With

$$\sum_f m_f^2 = 62.1 \text{ GeV}^2$$

then

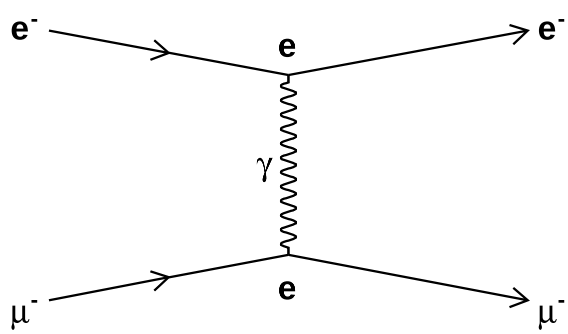
$$\mathcal{B}(b\bar{b}) = \frac{3m_b^2}{\sum_f m_f^2} = 0.852$$

Answer for experimental identification should include

- Higgs is predominantly decaying to b-quarks, so need to identify these to separate from backgrounds involving light quarks.
- B-quarks have non negligible lifetime, so can travel far enough to be identified via tracks with a large impact parameter or a secondary vertex.
- Reconstruct the Higgs mass from the jet masses.
- Can use the Z decay products to help reject backgrounds, in particular two electrons / muons or missing energy totalling the Z mass.

(iv) A Higgs of 250GeV will predominantly decay to W or Z pairs. Thus easiest way to identify these will be through their leptonic decay modes, particularly at the hadron colliders. ZZ decay to four muons is the gold-plated discovery channel for a higgs of this mass at LHC.

4. (i) The Feynman diagram for $e^- \mu^- \to e^- \mu^-$ scattering is



There are two vertices in the diagram, each with a power of e, so the amplitude is proportional to e^2 and hence the cross section to e^4 or α^2, as given.

(ii) Since the reaction is elastic, the electron energy E does not change. By scattering through an angle θ, then the momentum changes by $p(1 - \cos\theta)$ along the initial electron direction and $p \sin\theta$ perpendicular to it. Hence, the change in the four-momentum is

$$\begin{aligned} q^2 &= -p^2 \sin^2\theta - p^2(1 - \cos\theta)^2 \\ &= -p^2 \sin^2\theta - p^2 - p^2 \cos^2\theta + 2p^2 \cos\theta = -2p^2(1 - \cos\theta) \end{aligned}$$

In the centre-of-mass, the muon momentum is also p and, neglecting masses, then the square of the centre-of-mass energy $s = 4p^2$, so

$$q^2 = -\frac{1}{2}s(1 - \cos\theta)$$

Hence

$$\frac{d}{d(q^2)} = \frac{2}{s}\frac{d}{d(\cos\theta)}$$

Also, using $\cos\theta = 1 - 2\sin^2(\theta/2)$, then

$$q^2 = -s \sin^2(\theta/2)$$

or

$$\sin^2(\theta/2) = -\frac{q^2}{s}$$

so that

$$\cos^2(\theta/2) = 1 - \sin^2(\theta/2) = 1 + \frac{q^2}{s}$$

Therefore the cross section becomes

$$\begin{aligned}\frac{d\sigma}{d(q^2)} &= \frac{2}{s}\frac{d\sigma}{d(\cos\theta)} = \frac{2}{s}\frac{\pi\alpha^2}{s}\frac{1+\left[1+(q^2/s)\right]^2}{(q^4/s^2)} \\ &= \frac{2\pi\alpha^2}{q^4}\left[1+1+2\frac{q^2}{s}+\frac{q^4}{s^2}\right] \\ &= \frac{2\pi\alpha^2}{q^4}\left[\frac{q^4}{s^2}+2\left(1+\frac{q^2}{s}\right)\right]\end{aligned}$$

(iii) For a quark with momentum fraction x, then in the ep centre-of-mass, the total eq energy and momentum are

$$E_{eq} = p + xp = (1+x)p, \qquad P_{eq} = p - xp = (1-x)p$$

so the eq centre-of-mass energy is

$$\begin{aligned}\hat{s} &= E_{eq}^2 - P_{eq}^2 = (1+x)^2p^2 - (1-x)^2p^2 \\ &= (1+2x+x^2-1+2x-x^2)p^2 = 4xp^2 = xs\end{aligned}$$

(iv) The cross section for scattering from quark type i with fractional momentum x is

$$\frac{d\sigma}{d(q^2)} = \frac{2\pi\alpha^2}{q^4}\left(\frac{e_i}{e}\right)^2\left[\frac{q^4}{\hat{s}^2}+2\left(1+\frac{q^2}{\hat{s}}\right)\right] = \frac{2\pi\alpha^2}{q^4}\left(\frac{e_i}{e}\right)^2\left[\frac{q^4}{x^2s^2}+2\left(1+\frac{q^2}{xs}\right)\right]$$

so the total cross section is

$$\frac{d\sigma}{d(q^2)} = \frac{2\pi\alpha^2}{q^4}\sum_i\left(\frac{e_i}{e}\right)^2 p_i(x)\left[\frac{q^4}{x^2s^2}+2\left(1+\frac{q^2}{xs}\right)\right]dx$$

or

$$\frac{d\sigma}{dx\,d(q^2)} = \frac{2\pi\alpha^2}{q^4}\sum_i\left(\frac{e_i}{e}\right)^2 p_i(x)\left[\frac{q^4}{x^2s^2}+2\left(1+\frac{q^2}{xs}\right)\right]$$

Comparing with the conventional expression for the cross section, then

$$F_1(x,q^2) = \sum_i\left(\frac{e_i}{e}\right)^2 p_i(x), \qquad \frac{F_2(x,q^2)}{x} = 2\sum_i\left(\frac{e_i}{e}\right)^2 p_i(x)$$

Hence, the quark model predicts the structure functions are related by

$$F_2(x,q^2) = 2xF_1(x,q^2)$$

5. (a) The oscillations are due to a "beat" effect of having different frequencies. With all neutrinos having zero mass, all the phases would remain equal at all times.

(b) Solving the two equations for ν_μ and ν_τ, then

$$\nu_\mu \cos\theta = \nu_1 \cos^2\theta - \nu_2 \sin\theta\cos\theta, \qquad \nu_\tau \sin\theta = \nu_1 \sin^2\theta + \nu_2 \sin\theta\cos\theta$$

so

$$\nu_1 = \nu_\mu \cos\theta + \nu_\tau \sin\theta$$

Similarly

$$\nu_\mu \sin\theta = \nu_1 \sin\theta\cos\theta - \nu_2 \sin^2\theta, \qquad \nu_\tau \cos\theta = \nu_1 \sin\theta\cos\theta + \nu_2 \cos^2\theta$$

so

$$\nu_2 = -\nu_\mu \sin\theta + \nu_\tau \cos\theta$$

An initially pure muon neutrino beam at time $t = 0$ is in a state

$$\psi(0) = \nu_\mu = \nu_1 \cos\theta - \nu_2 \sin\theta$$

Each of the states ν_i change with time according to the standard quantum mechanical time dependence $e^{-E_i t}$, so at a later time t, the state is

$$\psi(t) = \nu_1 e^{-E_1 t}\cos\theta - \nu_2 e^{-E_2 t}\sin\theta$$

Substituting for ν_1 and ν_2, then this is

$$\begin{aligned}\psi(t) &= (\nu_\mu \cos\theta + \nu_\tau \sin\theta)e^{-E_1 t}\cos\theta - (-\nu_\mu \sin\theta + \nu_\tau \cos\theta)e^{-E_2 t}\sin\theta \\ &= \nu_\mu(e^{-E_1 t}\cos^2\theta + e^{-E_2 t}\sin^2\theta) + \nu_\tau(e^{-E_1 t}\sin\theta\cos\theta - e^{-E_2 t}\sin\theta\cos\theta)\end{aligned}$$

so the amplitude for muon neutrinos is

$$A_\mu = e^{-iE_1 t}\cos^2\theta + e^{-iE_2 t}\sin^2\theta$$

and for tau neutrinos is

$$A_\tau = \cos\theta\sin\theta\left(e^{-iE_2 t} - e^{-iE_1 t}\right)$$

(c) Writing

$$E_2 = \frac{E_2 + E_1}{2} + \frac{E_2 - E_1}{2}, \qquad E_1 = \frac{E_2 + E_1}{2} - \frac{E_2 - E_1}{2}$$

then the amplitude for having a tau neutrino in the beam is

$$\begin{aligned}A_\tau &= \cos\theta\sin\theta\left[e^{-i(E_2+E_1)t/2}e^{-i(E_2-E_1)t/2} - e^{-i(E_2+E_1)t/2}e^{i(E_2-E_1)t/2}\right] \\ &= \frac{1}{2}\sin 2\theta e^{-i(E_2+E_1)t/2}\left[e^{-i(E_2-E_1)t/2} - e^{i(E_2-E_1)t/2}\right] \\ &= -i\frac{1}{2}\sin 2\theta e^{-i(E_2+E_1)t/2}\sin\left[(E_2 - E_1)t/2\right]\end{aligned}$$

Hence, the probability of having a tau neutrino is

$$P_\tau = |A_\tau|^2 = \sin^2 2\theta \sin^2\left[\frac{(E_2 - E_1)t}{2}\right]$$

The general equation for the energy and momentum is

$$E = \sqrt{p^2 + m^2} = \left(p^2 + m^2\right)^{1/2}$$

If E is much greater than m, then so is p, so approximating the square root by a binomial expansion gives

$$E = p\left(1 + \frac{m^2}{p^2}\right)^{1/2} \approx p\left(1 + \frac{1}{2}\frac{m^2}{p^2}\right) = p + \frac{m^2}{2p}$$

Hence $E \approx p$ and

$$E_2 - E_1 \approx \frac{m_2^2 - m_1^2}{2p} = \frac{\Delta(m^2)}{2p} \approx \frac{\Delta(m^2)}{2E}$$

The distance gone is $l = \beta t$, but as the energy is much greater than the mass, then the velocity is effectively $\beta = 1$, so

$$P_\tau \approx \sin^2 2\theta \sin^2\left[\frac{\Delta(m^2)l}{4E}\right]^2$$

(d) The cosmic rays interact strongly in the atmosphere and so produce many pions. Of these, the charged pions then mostly decay to a muon and a muon neutrino

$$\pi^- \to \mu^-\overline{\nu}_\mu$$

and its charge conjugate. The muons themselves subsequently decay to an electron, an electron neutrino and another muon neutrino

$$\mu^- \to e^-\overline{\nu}_e\nu_\mu$$

and its charge conjugate. Hence, every charged pion results in two muon neutrinos and one electron neutrino.

Under the hypothesis of maximal mixing, the muon neutrinos have mixed to (undetected) tau neutrinos and so the ratio of muon to electron neutrinos can be less than two. The probability of a muon neutrino remaining in the beam is $P_\mu = 1 - P_\tau = R/2$, where R is the muon to electron neutrino ratio. For neutrinos coming vertically downwards, the limit on P_τ is therefore given by

$$P_\tau < 1 - 0.5R = 0.1$$

For maximal mixing, $\sin^2 2\theta = 1$, so

$$\sin^2\left[\frac{\Delta(m^2)l}{4E}\right] < 0.1$$

or

$$\frac{\Delta(m^2)l}{4E} < 0.32 \text{ rad}$$

Hence, with $E = 0.5$ GeV and l the thickness of the atmosphere = 20 km or 1.01×10^{20} GeV^{-1}, then

$$\Delta(m^2) < 6 \times 10^{-21} \text{ GeV}^2 = 6 \times 10^{-3} \text{ eV}^2$$

For neutrinos coming vertically upwards, then the $\sin^2$ term must be averaging to 0.5 as the ratio is independent of the angle (and hence path length) and energy. Hence,

$$\frac{\Delta(m^2)l}{4E} \gg 2\pi \text{ rad}$$

so with l the diameter of the Earth = 12800 km or 6.5×10^{22} GeV^{-1}, then

$$\Delta(m^2) \gg 2 \times 10^{-22} \text{ GeV}^2 = 2 \times 10^{-4} \text{ eV}^2$$

These set the limits on $\Delta(m^2)$.