Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 6

1. The SU(2)×U(1) local gauge theory of the electroweak interactions works without needing the Higgs if all boson and fermion masses are zero; this problem goes through this situation.

   Consider the case in which the electron is taken to be massless, as is the electron neutrino. Writing the doublet column vector

   $$\Psi = \begin{pmatrix} \psi_\nu \\ \psi_e \end{pmatrix}$$

   then write down the free Lagrangian density for these two particles in terms of Ψ. Transforming Ψ using the general 2×2 special unitary matrix

   $$U = e^{-i\alpha_i \sigma_i/2}$$

   for $\alpha_i \neq \alpha_i(x^\mu)$, then find the three Nöther currents, J^μ_{Vi}, associated with the global gauge invariance of the α_i's. Write these out explicitly in terms of ψ_e and ψ_ν.

   Because the fermions are massless then, as shown in Question 3 on Problem Sheet 5, the axial currents of the form $\overline{\psi}\gamma^\mu\gamma^5\psi$ are also conserved; hence there are three other conserved currents, J^μ_{Ai}. In addition, any mixture of these with the vector currents is therefore also conserved. The weak interactions use the (arbitrary) choice

   $$J^\mu_i = \frac{1}{2}\left(J^\mu_{Vi} - J^\mu_{Ai}\right)$$

   Write down the interaction terms expected from local gauge invariance for the two currents J^μ_1 and J^μ_2 coupling to two fields, here called W^μ_1 and W^μ_2. Defining two combinations of these fields as

   $$W^{\pm\mu} = \frac{1}{\sqrt{2}}\left(W^\mu_1 \pm iW^\mu_2\right)$$

   write the combined interaction in terms of the $W^{\pm\mu}$ fields, again in terms of ψ_e and ψ_ν.

   The third field W^μ_3 mixes with the "hypercharge" field Y^μ to give the physical Z and photon particles

   $$Z^\mu = W^\mu_3 \cos\theta_W - Y^\mu \sin\theta_W, \qquad A^\mu = W^\mu_3 \sin\theta_W + Y^\mu \cos\theta_W$$

   Show that, if the hypercharge field couples to the electron and neutrino, then the total neutral interaction term is

   $$g_W W_{3\mu} J^\mu_3 + \frac{g_Y}{2} Y_\mu J^\mu_Y$$

   and the hypercharge current is taken as the most general form

   $$J^\mu_Y = a\overline{\psi}_{\nu L}\gamma^\mu\psi_{\nu L} + b\overline{\psi}_{\nu R}\gamma^\mu\psi_{\nu R} + c\overline{\psi}_{eL}\gamma^\mu\psi_{eL} + d\overline{\psi}_{eR}\gamma^\mu\psi_{eR}$$

   then the correct photon coupling is given if

   $$g_W \sin\theta_W = g_Y \cos\theta_W, \qquad e = g_W \sin\theta_W$$

   and

   $$a = -1, \qquad b = 0, \qquad c = -1, \qquad d = -2$$

The general form for a coupling of the Z to fermions is

$$\frac{1}{2} g_Z Z_\mu \overline{\psi}_f \gamma^\mu (c_{fV} - c_{fA}\gamma^5)\psi_f$$

Show the couplings of both the neutrino and the electron to the Z have a strength

$$g_Z = \frac{g_W}{\cos\theta_W}$$

and the vector and axial couplings are

$$c_{\nu V} = \frac{1}{2}, \qquad c_{\nu A} = \frac{1}{2}, \qquad c_{eV} = -\frac{1}{2} + 2\sin^2\theta_W, \qquad c_{eA} = -\frac{1}{2}$$

2. This question works through the calculation of the Z decay widths and branching ratios.

   The spin-averaged matrix element of the decay $Z \to f\overline{f}$ for any fermion f can be calculated from the Feynman diagram to be

   $$\langle |M|^2 \rangle = \frac{g_W^2 M_Z^2}{3\cos^2\theta_W}(c_{fV}^2 + c_{fA}^2)$$

   where the fermion mass has been neglected. Show that the partial width for this mode is therefore

   $$\Gamma(Z^0 \to f\overline{f}) = \frac{G_F}{\sqrt{2}} \frac{M_Z^3}{6\pi} \left(c_{fV}^2 + c_{fA}^2\right)$$

   Evaluate this for the case of $f = \nu_e$.

   - List the possible decay modes for the Z^0 to leptons or quarks
   - Calculate the partial width for the other lepton decays and for decays to hadrons, neglecting all fermion masses and higher-order QCD effects.
   - Calculate the visible partial width. What was this, in conjunction with the total width, used to determine?
   - Calculate the total width and hence lifetime of the Z^0 and find the leptonic and hadronic branching ratios. What fractions of the hadronic decays contain charmed and bottom hadrons? Compare these numbers with those for the $W^\pm$ decays in Question 3 on Problem Sheet 6.

3. Exam question: 2004 question 3.

   The search for the Higgs boson is one of the major goals of the worldwide particle physics programme. In the following, you may take the mass of the W to be 80 GeV, the mass of the Z to be 91 GeV, the quark masses to be; u 1 MeV, d 2 MeV, s 200 MeV, c 1.4 GeV, b 4.2 GeV and t 175 GeV, the charged lepton masses to be; e 0.51 MeV, μ 106 MeV and τ 1.78 GeV and the neutrinos to be massless.

   (i) How do the coupling of the Higgs to a fermion and the associated rate for that process depend on the mass of the fermion?

   (ii) LEP2 was an electron-positron collider at CERN working at centre-of-mass energies, E_{cm}, up to 209 GeV. Give the Feynman diagram for the principal Higgs production mechanism at LEP2.

This mechanism sets a kinematic limit on the mass of the Higgs which could be produced. At a centre-of-mass energy of 209 GeV, estimate this limit. For Higgs masses less than this limit, show that the energy of the Higgs produced is given by

$$E_H = \frac{E_{cm}^2 + m_H^2 - m_Z^2}{2E_{cm}}$$

Evaluate this for the case of a Higgs with mass of 115 GeV. Hence, discuss the realistic limit on the Higgs mass which could be set through direct searches at LEP2.

(iii) What is the dominant decay mode of a Standard Model Higgs with a mass around 115 GeV? Neglecting any differences in phase space, estimate the branching fraction for the dominant mode. Discuss how you would identify this decay experimentally.

(iv) Discuss qualitatively how your answers to part (iii) above would differ for a Higgs of mass around 250GeV.

4. Exam question: 2002 question 1.

(i) Consider the QED elastic scattering reaction $e^-\mu^- \to e^-\mu^-$. Neglecting all masses, the cross section is given by

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{s}\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}$$

where α is the fine structure constant, s is the square of the centre-of-mass energy and θ is the angle of the scattered electron in the $e\mu$ centre-of-mass frame. Draw a Feynman diagram for this reaction and justify the power of α in the above expression.

(ii) Show that the change of the electron four-momentum, q^μ, satisfies

$$q^2 = q^\mu q_\mu = -\frac{1}{2}s(1-\cos\theta)$$

Hence show the cross section as a function of q^2 is

$$\frac{d\sigma}{d(q^2)} = \frac{2\pi\alpha^2}{q^4}\left[\frac{q^4}{s^2} + 2\left(1+\frac{q^2}{s}\right)\right]$$

(iii) Deep inelastic scattering of electrons from protons can be described by the reaction $e^-p \to e^-X$, where X is a hadronic system with $m_X \gg m_p$. Within the quark model, this can be considered to arise from elastic electron-quark scattering. If the scattered quark initially had a fraction x of the proton momentum then, neglecting masses and transverse momentum components, show that the centre-of-mass energy of the electron-quark system $\hat{s}$ is related to that of the electron-proton system s by

$$\hat{s} = xs$$

(iv) If the probability of each type of quark i ($i = u, d, s, \ldots$) having a momentum fraction x is $p_i(x)dx$, then using the results from (ii) and (iii), write down the cross section as a function of q^2 and x for the electron-proton reaction from the quark model. Conventionally, the cross section for the electron-proton reaction has been written as

$$\frac{d\sigma}{dx\,d(q^2)} = \frac{2\pi\alpha^2}{q^4}\left[\left(\frac{q^2}{xs}\right)^2 F_1(x,q^2) + \left(1+\frac{q^2}{xs}\right)\frac{F_2(x,q^2)}{x}\right]$$

in terms of two arbitrary "structure functions" F_1 and F_2. Hence deduce a relation between F_1 and F_2.

5. Exam question: 2001 question 8, non-essay parts of guest lecture based question.

(a) Explain why the observation of neutrino oscillations implies that at least some neutrino types must have a non-zero mass.

(b) Consider the case of muon neutrino and tau neutrino mixing. Each is a superposition of two states, ν_1 and ν_2, of definite masses, m_1 and m_2, such that

$$\nu_\mu = \nu_1 \cos\theta - \nu_2 \sin\theta, \qquad \nu_\tau = \nu_1 \sin\theta + \nu_2 \cos\theta.$$

A pure muon neutrino beam of momentum p is created at time $t = 0$. What is the composition of the neutrino state at a later time t? Show that the amplitude of muon neutrinos in the beam is given by

$$A_\mu = e^{-iE_1 t} \cos^2\theta + e^{-iE_2 t} \sin^2\theta,$$

and for tau neutrinos is given by

$$A_\tau = \cos\theta \sin\theta \left(e^{-iE_2 t} - e^{-iE_1 t} \right),$$

where E_i is the energy of the state ν_i.

(c) Show that the probability of observing a tau neutrino at time t is

$$P_\tau = \sin^2 2\theta \sin^2 \left[\frac{(E_2 - E_1)t}{2} \right].$$

Assuming that $E_i \gg m_i$, show that the probability of observing a tau neutrino at a distance l from the source is

$$P_\tau \approx \sin^2 2\theta \sin^2 \left[\frac{\Delta(m^2) l}{4E} \right],$$

where $\Delta(m^2)$ is the difference of the squares of the masses.

(d) Atmospheric neutrinos are produced by cosmic rays interacting in the upper atmosphere. Explain why the expected ratio of the number of muon (anti)neutrinos to the number of electron (anti)neutrinos from the atmosphere is two. The SuperK experiment measures this ratio for neutrinos coming vertically downwards and obtains a value consistent with two, with a lower limit of 1.8. However, for the neutrinos coming vertically upwards, a value approximately half that expected is obtained. This value is found to be independent of the measured angle or energy of the neutrinos. Under the hypothesis of maximal muon/tau neutrino mixing, such that $\theta = 45°$, what what can be deduced about $\Delta(m^2)$? The average energy of the detected neutrinos should be taken as 0.5 GeV. The radius of the Earth is 6400 km and the thickness of the atmosphere is 20 km.