Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 5 Answers

1. The required values for P and C needed for this question are that the ground state mesons are $J^P = 0^-$ with the π^0, η and η' being $J^{PC} = 0^{-+}$, while the first excited state mesons are $J^P = 1^-$ with the ρ^0, ω and ϕ being $J^{PC} = 1^{--}$, as listed on the handout.

   These can also be obtained from the following. Any state of orbital angular momentum L has a parity for the spatial part of the wavefunction of $P_L = (-1)^L$. If the state is made of a particle-antiparticle pair, then the charge conjugation operation changes each one into the other and a subsequent parity operation returns the state to the original one. Hence, the C value must be the same as for P, namely $C_L = (-1)^L$.

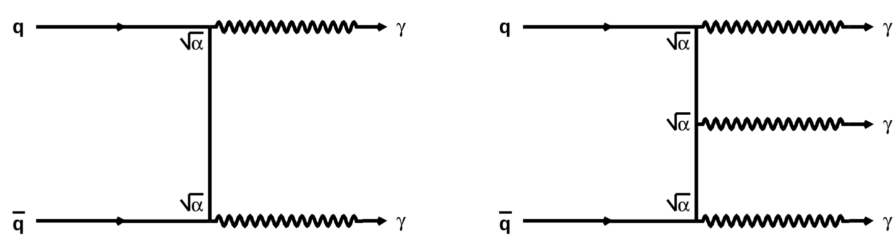
   Hence, any fermion-antifermion pair has an overall parity $P = (-1)^{L+1}$. A state with the same type of fermion-antifermion is an eigenstate of C also, with a value $C = (-1)^{L+S}$, where the other factors come from the spin and intrinsic C values.

   The photon has an intrinsic parity of $P = -1$ as the A^μ field is a polar (true) vector and the spatial components change sign under parity. The photon value of $C_\gamma = -1$ as the electromagnetic fields and potentials change sign under a charge conjugation operation on all the charges in a system. Hence, the photon is $J^{PC} = 1^{--}$, which is the same as the excited state mesons.

   There is also an issue of either Pauli or Bose-Einstein symmetries if there are identical fermions or bosons, respectively, in the final state.

   Taking each decay type in turn;

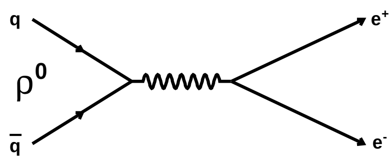
   - For the photon decays, then in the absence of any conservation law, the expectation would be that the three photon decay would be approximately a factor of $\alpha = 1/137$ lower than the two photon decay. This is clearly not the case.

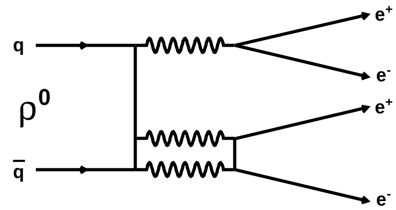
     

     The C value of all the initial states is $C = +1$. The C value of a state of n photons is $C_{n\gamma} = (-1)^n$ and so to conserve C, then n has to be even. Therefore, the two photon decays are allowed but three photon decays are forbidden.

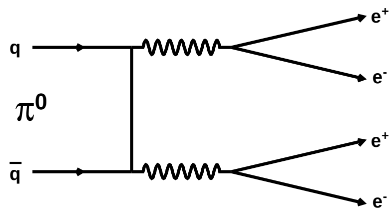
   - For the electron-positron decays, again in the absence of any conservation law, then the $e^+e^-e^+e^-$ decays would be expected to be a factor α^2 less than the e^+e^- decays. In addition, the ρ^0 and π^0 decays would be expected to be roughly the same magnitude.

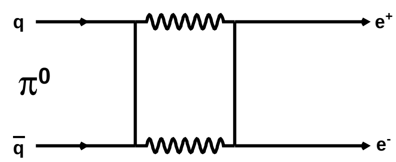
     The decays are via virtual photons; as above the π^0 must decay via an even number of photons, while the ρ^0, with $C = -1$, must decay through an odd number. Hence, the ρ^0 can decay through a single virtual photon to e^+e^- and so is proportional to α^2. The highest diagram for $\rho^0 \to e^+e^-e^+e^-$ involves three photons and is therefore α^6 and so is heavily suppressed.





However the π^0 must decay via two photons, either to $e^+e^-e^+e^-$ which is α^4, or possibly to e^+e^- through a box diagram, which has the same number of vertices and hence power of α^4. Hence, the $\pi^0 \to e^+e^-$ decay is not absolutely forbidden, but just suppressed.





- Consider the decay $\eta \to \pi^+\pi^-$. The initial parity is due to the η and so is $P = -1$. Parity due to orbital angular momentum is $(-1)^L$ so the total parity of the final state is $(-1)^L(-1)(-1) = (-1)^L$. However, as all the particles involved have spin-0, then to conserve angular momentum, then $L = 0$ also. Hence, the parity of the final state is $P = +1$. This decay is therefore forbidden by parity conservation. This argument holds for any of the ground state mesons decaying to any other two and so is true for the η' also. In contrast, the $\pi^+\pi^-\pi^0$ final state can have any value of L between (for example) the $\pi^+\pi^-$ but then must also have the same L between this pair and the π^0 so as to be able to combine two orbital angular momenta to give zero overall. Hence, the total parity is $(-1)^L(-1)^L(-1)(-1)(-1) = -1$ and this is allowed by parity conservation. The $\pi^0\pi^0\pi^0$ state is very similar except Bose-Einstein symmetry requires the first $\pi^0\pi^0$ pair to be in an even L state, so only even values of L are allowed. However, the overall parity is still $P = -1$.

  For the ρ^0, the initial state is again $P = -1$ but it now has $J = 1$. Hence, for two pions $L = 1$ is required and so the final state parity for two pions is $(-1)^L(-1)(-1) = -1$. This means $\rho^0 \to \pi^+\pi^-$ is allowed by parity conservation. Again, the same argument holds for any first excited state meson decaying to two ground state mesons and so $\rho^\pm \to \pi^\pm\pi^0$ is also allowed. Note, the ρ^0 decay must also obey C conservation, whereas there is no defined C value for the $\rho^\pm$ as it is not a C eigenstate. For a $\pi^+\pi^-$ state, then $C = (-1)^L = -1$ which is the same as the ρ^0 and so the decay is allowed by C conservation also.

  However, the decays $\rho^0 \to \pi^0\pi^0$ and $\rho^0 \to \eta\pi^0$ are not. Since the final state particles are C eigenstates, then a C operation simply gives a factor of $C_{\pi^0}^2 = +1$ or $C_{\pi^0}C_\eta = +1$ respectively as no further exchange of the particles is required to obtain the original state. Hence, these decays are forbidden by C conservation. Note, the $\rho^0 \to \pi^0\pi^0$ is also forbidden by P conservation, as the Bose-Einstein symmetry requires that the two identical bosons are in an even L state.

2. Applying the operator combination to Ψ gives

$$(\lambda_1 - \hat{A})\Psi = \sum_i \alpha_i(\lambda_1 - \hat{A})\psi_i$$

But, by definition

$$\hat{A}\psi_i = \lambda_i\psi_i$$

so

$$(\lambda_1 - \hat{A})\Psi = \sum_i \alpha_i(\lambda_1 - \lambda_i)\psi_i$$

Hence, the term containing ψ_1 is multiplied by zero and so is removed. All the other coefficients get an additional factor $\lambda_1 - \lambda_i$. With only two eigenvalues, then this becomes

$$(\lambda_1 - \hat{A})\Psi = \alpha_2(\lambda_1 - \lambda_2)\psi_2$$

so that

$$\frac{\lambda_1 - \hat{A}}{\lambda_1 - \lambda_2}\Psi = P_2\Psi = \alpha_2\psi_2$$

Thus P_2 removes ψ_1 and leaves the coefficient of ψ_2 unchanged. Similarly

$$\frac{\lambda_2 - \hat{A}}{\lambda_2 - \lambda_1}\Psi = P_1\Psi = \alpha_1\psi_1$$

The definition of $\hat{B}$ is

$$\hat{B} = \frac{2\hat{A} - (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)}$$

Therefore

$$\hat{B}\psi_1 = \frac{2\hat{A}\psi_1 - (\lambda_1 + \lambda_2)\psi_1}{(\lambda_1 - \lambda_2)} = \frac{2\lambda_1\psi_1 - (\lambda_1 + \lambda_2)\psi_1}{(\lambda_1 - \lambda_2)} = \frac{\lambda_1\psi_1 - \lambda_2\psi_1}{(\lambda_1 - \lambda_2)} = \psi_1$$

and similarly

$$\hat{B}\psi_2 = \frac{2\hat{A}\psi_2 - (\lambda_1 + \lambda_2)\psi_2}{(\lambda_1 - \lambda_2)} = \frac{2\lambda_2\psi_2 - (\lambda_1 + \lambda_2)\psi_2}{(\lambda_1 - \lambda_2)} = \frac{\lambda_2\psi_2 - \lambda_1\psi_2}{(\lambda_1 - \lambda_2)} = -\psi_2$$

Hence, $\hat{B}$ has the same eigenstates as $\hat{A}$, but with eigenvalues of ± 1. Also consider

$$\hat{B}^2\Psi = \hat{B}\hat{B}(\alpha_1\psi_1 + \alpha_2\psi_2) = \hat{B}(\alpha_1\psi_1 - \alpha_2\psi_2) = \alpha_1\psi_1 + \alpha_2\psi_2 = \Psi$$

Since this holds for any state Ψ, then $\hat{B}^2 = 1$.
Solving for $\hat{A}$ in terms of $\hat{B}$ gives

$$\hat{A} = \frac{1}{2}\left[(\lambda_1 + \lambda_2) + \hat{B}(\lambda_1 - \lambda_2)\right]$$

Hence

$$\hat{P}_1 = \frac{\lambda_2 - \hat{A}}{\lambda_2 - \lambda_1} = \frac{(\lambda_2 - \lambda_1) - \hat{B}(\lambda_1 - \lambda_2)}{2(\lambda_2 - \lambda_1)} = \frac{1}{2}(1 + \hat{B})$$

and similarly

$$\hat{P}_2 = \frac{\lambda_1 - \hat{A}}{\lambda_1 - \lambda_2} = \frac{(\lambda_1 - \lambda_2) - \hat{B}(\lambda_1 - \lambda_2)}{2(\lambda_1 - \lambda_2)} = \frac{1}{2}(1 - \hat{B})$$

Cross check that these do indeed act as projection operators

$$P_1\Psi = \frac{1}{2}(1+\hat{B})(\alpha_1\psi_1 + \alpha_2\psi_2) = \frac{1}{2}\left[(\alpha_1\psi_1 + \alpha_2\psi_2) + (\alpha_1\psi_1 - \alpha_2\psi_2)\right] = \alpha_1\psi_1$$

and similarly

$$P_2\Psi = \frac{1}{2}(1-\hat{B})(\alpha_1\psi_1 + \alpha_2\psi_2) = \frac{1}{2}\left[(\alpha_1\psi_1 + \alpha_2\psi_2) - (\alpha_1\psi_1 - \alpha_2\psi_2)\right] = \alpha_2\psi_2$$

and so $P_{1,2}$ do act as projection operators.

The other properties follow as

$$P_1^2 = \frac{1}{4}(1+\hat{B})(1+\hat{B}) = \frac{1}{4}(1+2\hat{B}+\hat{B}^2) = \frac{1}{2}(1+\hat{B}) = P_1$$

Similarly

$$P_2^2 = \frac{1}{4}(1-\hat{B})(1-\hat{B}) = \frac{1}{4}(1-2\hat{B}+\hat{B}^2) = \frac{1}{2}(1-\hat{B}) = P_2$$

Also

$$P_1P_2 = \frac{1}{4}(1+\hat{B})(1-\hat{B}) = \frac{1}{4}(1-\hat{B}^2) = 0$$

as does

$$P_2P_1 = \frac{1}{4}(1-\hat{B})(1+\hat{B}) = \frac{1}{4}(1-\hat{B}^2) = 0$$

Finally

$$P_1 + P_2 = \frac{1}{2}\left[(1+\hat{B}) + (1-\hat{B})\right] = 1$$

3. The handedness eigenvalue equation is

$$\frac{1}{2}\gamma^5\psi = \lambda\psi$$

Applying the handedness operator again gives

$$\frac{1}{4}\gamma^5\gamma^5\psi = \lambda\frac{1}{2}\gamma^5\psi = \lambda^2\psi$$

Since $\gamma^5\gamma^5 = 1$, then this means

$$\lambda^2 = \frac{1}{4} \quad \text{so} \quad \lambda = \pm\frac{1}{2}$$

With $\hat{A} = \gamma^5/2$, then taking $\lambda_1 = 1/2$ and $\lambda_2 = -1/2$, then

$$\hat{B} = \frac{2\hat{A} - (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)} = \frac{\gamma^5 - 0}{1} = \gamma^5$$

Hence

$$P_1 = \frac{1}{2}(1+\hat{B}) = \frac{1}{2}(1+\gamma^5), \qquad P_2 = \frac{1}{2}(1-\hat{B}) = \frac{1}{2}(1-\gamma^5)$$

as expected.

The Dirac Hamiltonian is

$$\hat{H} = -i\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla} + m\gamma^0$$

Hence, the commutator with the handedness operator is

$$\begin{aligned}\left[\hat{H},\frac{1}{2}\gamma^5\right] &= -i\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla},\frac{1}{2}\gamma^5\right]+m\left[\gamma^0,\frac{1}{2}\gamma^5\right]\\ &= -\frac{i}{2}\left(\gamma^0\boldsymbol{\gamma}\gamma^5-\gamma^5\gamma^0\boldsymbol{\gamma}\right).\boldsymbol{\nabla}+\frac{m}{2}\left(\gamma^0\gamma^5-\gamma^5\gamma^0\right)\end{aligned}$$

But, since γ^5 anticommutes with all the γ^μ

$$\{\gamma^\mu,\gamma^5\}=\gamma^\mu\gamma^5+\gamma^5\gamma^\mu=0$$

then switching γ^5 and any γ^μ introduces a minus sign, so

$$\gamma^5\gamma^0=-\gamma^0\gamma^5$$

and

$$\gamma^5\gamma^0\boldsymbol{\gamma}=-\gamma^0\gamma^5\boldsymbol{\gamma}=\gamma^0\boldsymbol{\gamma}\gamma^5$$

Hence

$$\left[\hat{H},\frac{1}{2}\gamma^5\right]=m\gamma^0\gamma^5$$

and hence does not commute.

The Dirac equation is

$$i\gamma^\mu\partial_\mu\psi=m\psi$$

or

$$\gamma^\mu\partial_\mu\psi=-im\psi$$

Hence, taking the Hermitian conjugate

$$(\partial_\mu\psi^\dagger)\gamma^{\mu\dagger}=im\psi^\dagger$$

and using

$$\gamma^{\mu\dagger}=\gamma^0\gamma^\mu\gamma^0$$

then

$$(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0=im\psi^\dagger\gamma^0$$

so

$$(\partial_\mu\overline{\psi})\gamma^\mu=im\overline{\psi}$$

is the Dirac Hermitian conjugate equation. Taking the derivative of the γ^5 current gives

$$\begin{aligned}\partial_\mu(\overline{\psi}\gamma^\mu\gamma^5\psi) &= (\partial_\mu\overline{\psi})\gamma^\mu\gamma^5\psi+\overline{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi)\\ &= (\partial_\mu\overline{\psi})\gamma^\mu\gamma^5\psi-\overline{\psi}\gamma^5\gamma^\mu(\partial_\mu\psi)\\ &= (im\overline{\psi})\gamma^5\psi-\overline{\psi}\gamma^5(-im\psi)\\ &= 2im\overline{\psi}\gamma^5\psi\end{aligned}$$

Hence, the current is not conserved. However, in the limit of zero mass, the Hamiltonian commutator becomes zero and the current becomes conserved.

Applying the handedness operator to each state in turn

$$\frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}a\\b\\a\\b\end{pmatrix}=\frac{1}{2}\begin{pmatrix}a\\b\\a\\b\end{pmatrix}$$

and

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ -a \\ -b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -a \\ -b \\ a \\ b \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} a \\ b \\ -a \\ -b \end{pmatrix}$$

Hence, these states are eigenvectors of handedness with eigenvalues of $\pm 1/2$ (or equivalently right-handed and left-handed), respectively.

The Dirac solutions are not of the form of these eigenvectors and so are not eigenstates of handedness. However, in the high energy limit or when the mass is zero, then $E \sim p \gg m$ and so $p/(E+m) \to p/E \to 1$. Hence, in this limit, the Dirac solutions go to

$$u_1 = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_2 = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

and so are also handedness eigenstates of eigenvalues $\pm 1/2$ respectively, which therefore correspond to helicity in this limit.

The explicit form of the $P_{1,2}$ operators in the standard representation is

$$P_{1,2} = \frac{1}{2}(1 \pm \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & \pm 1 \\ \pm 1 & 0 & 1 & 0 \\ 0 & \pm 1 & 0 & 1 \end{pmatrix}$$

so applying P_1 to u_1 gives

$$P_1 u_1 = a\psi_R = \frac{\sqrt{E+m}}{2}\begin{pmatrix} 1 + p/(E+m) \\ 0 \\ 1 + p/(E+m) \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} |a|^2 \psi_R^\dagger \psi_R &= \frac{E+m}{4}\begin{pmatrix} 1+p/(E+m) & 0 & 1+p/(E+m) & 0 \end{pmatrix}\begin{pmatrix} 1 + p/(E+m) \\ 0 \\ 1 + p/(E+m) \\ 0 \end{pmatrix} \\ &= \frac{E+m}{2}\left[1 + \frac{p}{E+m}\right]^2 = \frac{(E+m+p)^2}{2(E+m)} \end{aligned}$$

Assuming $\psi_R^\dagger \psi_R = 2E$, then

$$\begin{aligned} |a|^2 &= \frac{(E+m+p)^2}{4E(E+m)} = \frac{E^2 + m^2 + p^2 + 2Em + 2Ep + 2mp}{4E(E+m)} \\ &= \frac{2E(E+m) + 2p(E+m)}{4E(E+m)} = \frac{E+p}{2E} = \frac{1 + (p/E)}{2} = \frac{1+\beta}{2} \end{aligned}$$

Hence

$$|a| = \sqrt{\frac{1+\beta}{2}}$$

Similarly

$$P_2 u_1 = b\psi_L = \frac{\sqrt{E+m}}{2}\begin{pmatrix} 1-p/(E+m) \\ 0 \\ -1+p/(E+m) \\ 0 \end{pmatrix}$$

so

$$\begin{aligned} |b|^2\psi_R^\dagger\psi_R &= \frac{E+m}{4}\begin{pmatrix} 1-p/(E+m) & 0 & -1+p/(E+m) & 0 \end{pmatrix}\begin{pmatrix} 1-p/(E+m) \\ 0 \\ -1+p/(E+m) \\ 0 \end{pmatrix} \\ &= \frac{E+m}{2}\left[1-\frac{p}{E+m}\right]^2 = \frac{(E+m-p)^2}{2(E+m)} \end{aligned}$$

and so

$$\begin{aligned} |b|^2 &= \frac{(E+m-p)^2}{4E(E+m)} = \frac{E^2+m^2+p^2+2Em-2Ep-2mp}{4E(E+m)} \\ &= \frac{2E(E+m)-2p(E+m)}{4E(E+m)} = \frac{E-p}{2E} = \frac{1-(p/E)}{2} = \frac{1-\beta}{2} \end{aligned}$$

which gives

$$|b| = \sqrt{\frac{1-\beta}{2}}$$

Hence, the helicity $+1/2$ eigenstate has amplitudes of $\sqrt{(1\pm\beta)/2}$ of right- and left-handed states respectively.

4. The general formula for the partial width of a particle X is

$$\Gamma = \frac{|M|^2\rho}{2m_X}$$

The phase space for a two-body decay, neglecting masses, is

$$\frac{d\rho}{d\Omega} = \frac{1}{32\pi^2}$$

For the W^- decay to $e^-\overline{\nu}_e$, the matrix element is given as

$$\langle|M|^2\rangle = \frac{g_W^2 M_W^2}{3}$$

Since this is independent of solid angle, the integral over Ω can be done immediately to give

$$\rho = \frac{4\pi}{32\pi^2} = \frac{1}{8\pi}$$

Hence, the partial width for this decay is

$$\Gamma(W^- \to e^-\overline{\nu}_e) = \frac{g_W^2 M_W^2}{3}\frac{1}{8\pi}\frac{1}{2M_W} = \frac{g_W^2 M_W}{48\pi}$$

In terms of the Fermi constant, this is

$$\Gamma(W^- \to e^-\overline{\nu}_e) = \frac{g_W^2}{8M_W^2}\frac{M_W^3}{6\pi} = \frac{G_F}{\sqrt{2}}\frac{M_W^3}{6\pi}$$

With $M_W = 80.42$ GeV, this is 0.228 GeV.

The possible decay modes for the $W^\pm$ are to leptons, specifically $e\nu_e$, $\mu\nu_\mu$ and $\tau\nu_\tau$, or to quarks, specifically ud, us, ub, cd, cs and cb. No top quark decays are allowed as $m_t > M_W$. Neglecting the masses, then because of the universality of the weak coupling g_W, all the leptonic decays have the same rate as above, Γ_l. This is not true, however, for the quarks as the matrix element for ud, for example, will have an extra CKM matrix element factor V_{ud} and an extra factor of three because of the three colours of quarks, and so will be different from that for the leptons. The partial width to ud will therefore be

$$\Gamma_{ud} = 3|V_{ud}|^2\Gamma_l = 0.666 \text{ GeV}$$

Similarly, the other modes will have CKM factors; the total quark decay rate is therefore

$$\Gamma_q = 3\left(|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 + |V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2\right)\Gamma_l$$

However, the CKM matrix is unitary, meaning

$$VV^\dagger = I$$

which means

$$V_{ij}V^*_{kj} = \delta_{ik}$$

Hence, for $i = k = u$

$$V_{ud}V^*_{ud} + V_{us}V^*_{us} + V_{ub}V^*_{ub} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

and similarly for $i = k = c$, so

$$\Gamma_q = 6\Gamma_l = 1.365 \text{ GeV}$$

Hence, the total width is

$$\Gamma = 3\Gamma_l + \Gamma_q = 9\Gamma_l = 2.05 \text{ GeV}$$

and the lifetime is therefore

$$\tau = \frac{1}{\Gamma} = 3.2 \times 10^{-25} \text{ s}$$

The branching fractions to each lepton type should be

$$\mathcal{B}(W^- \to l^-\overline{\nu}_l) = \frac{\Gamma_l}{\Gamma} = \frac{1}{9} = 11.1\%$$

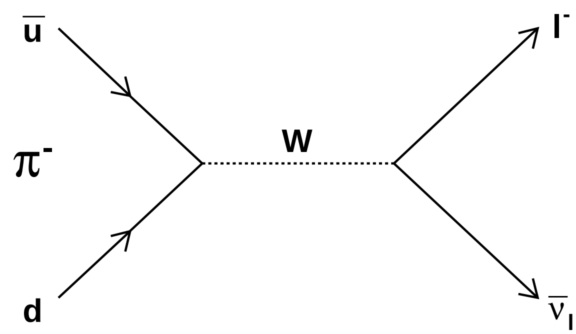
and to hadrons will be

$$\mathcal{B}(W^- \to \text{hadrons}) = \frac{\Gamma_q}{\Gamma} = \frac{6}{9} = 66.7\%$$

The fractions of the hadronic decays which contain c and b quarks is

$$R_c = \frac{|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2}{2} = 50\%, \qquad R_b \sim \frac{|V_{cb}|^2}{2} = 0.08\%$$

5. (i) The Feynman diagram for pion decay is



(ii) The matrix element depends on the physics of the interaction causing the decay. The phase space is a measure of the number of final states available for the decay and is independent of the way in which the interaction happens. The width is a measure of the uncertainty in the mass of the unstable particle and is inversely related to the lifetime.

(iii) The ratio of the magnitudes of the phase space for the two decays is

$$\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} = 2.34$$

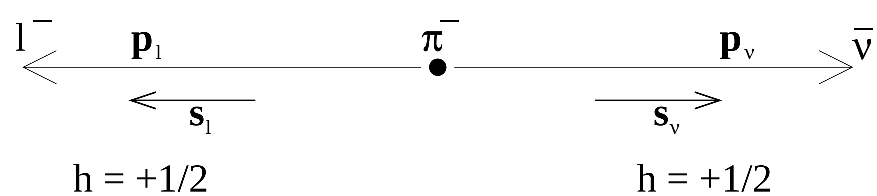
Hence, purely from the phase space, the electron decay would be expected to have the larger branching fraction by a factor of 2.34. Therefore, the strong suppression of the electron decay must come from the matrix element.

(iv) Combining the phase space and matrix elements, the ratio of the partial widths for the two decays is

$$\frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} = 1.28 \times 10^{-4}$$

which is in reasonable agreement with the measured value.

(v) Being an antiparticle, the antineutrino must have been produced in a right-handed state. Since it is assumed to be massless, and so has $\beta = 1$, then this has no helicity $-1/2$ component and the antineutrino must have helicity $+1/2$. Since the pion is spin 0, then to conserve angular momentum, the lepton must also be helicity $+1/2$, so the spins cancel, as shown below.



(vi) The leptons must be produced in the decay in left-handed states. However, from the diagram above, they must have helicity $+1/2$ to conserve angular momentum. Hence, there is a suppression factor of $1 - \beta_l$ in the rate due to projecting the left-handed state onto the helicity $+1/2$ state. This suppression factor would be zero for a particle with $\beta_l = 1$, i.e. a massless charged lepton and this explains the behaviour of the matrix element for $m_l = 0$. The electron is much lighter than the muon, so it has a much higher velocity leading to a heavier suppression and so the much lower electron branching fraction.

(vii) The charged kaon has new types of decays available to it as its mass is high enough that is able to decay hadronically to two or three pions or semileptonically to lepton, antineutrino and pions. Hence, with these extra channels, the purely leptonic decay modes do not completely dominate the decays.
The ratio of the electron to muon decays should be given by an equivalent expression to that in part (iv) above

$$\frac{m_e^2(m_K^2 - m_e^2)^2}{m_\mu^2(m_K^2 - m_\mu^2)^2} = 2.57 \times 10^{-5}$$

and so the electron decay mode would be expected to be 1.63×10^{-5}.

6. Using

$$K_S^0 = N\left(K_1^0 + \epsilon K_2^0\right), \qquad K_L^0 = N\left(K_2^0 + \epsilon K_1^0\right)$$

then

$$K_S^0 = \frac{N}{\sqrt{2}}\left[\left(K^0 + \overline{K}^0\right) + \epsilon\left(K^0 - \overline{K}^0\right)\right], \qquad K_L^0 = \frac{N}{\sqrt{2}}\left[\left(K^0 - \overline{K}^0\right) + \epsilon\left(K^0 + \overline{K}^0\right)\right]$$

so

$$K_S^0 = \frac{N}{\sqrt{2}}\left[(1+\epsilon)K^0 + (1-\epsilon)\overline{K}^0\right], \qquad K_L^0 = \frac{N}{\sqrt{2}}\left[(1+\epsilon)K^0 - (1-\epsilon)\overline{K}^0\right]$$

Adding and subtracting these, then

$$N\sqrt{2}(1+\epsilon)K^0 = K_S^0 + K_L^0, \qquad N\sqrt{2}(1-\epsilon)\overline{K}^0 = K_S^0 - K_L^0$$

so

$$K^0 = \frac{1}{N\sqrt{2}(1+\epsilon)}\left(K_S^0 + K_L^0\right), \qquad \overline{K}^0 = \frac{1}{N\sqrt{2}(1-\epsilon)}\left(K_S^0 - K_L^0\right)$$

An initially pure K^0 beam is in a state at time zero of

$$\psi(0) = \frac{1}{N\sqrt{2}(1+\epsilon)}\left(K_S^0 + K_L^0\right)$$

At a later time, the state is

$$\psi(t) = \frac{1}{N\sqrt{2}(1+\epsilon)}\left[f_S(t)K_S^0 + f_L(t)K_L^0\right]$$

where

$$f_S(t) = e^{-im_S t}e^{-\Gamma_S t/2}, \qquad f_L(t) = e^{-im_L t}e^{-\Gamma_L t/2}$$

To find the rate of semi-leptonic decays, $\psi(t)$ must be expressed in terms of K^0 and $\overline{K}^0$

$$\begin{aligned}\psi(t) &= \frac{1}{N\sqrt{2}(1+\epsilon)}\frac{N}{\sqrt{2}}\left[f_S(1+\epsilon)K^0 + f_S(1-\epsilon)\overline{K}^0 + f_L(1+\epsilon)K^0 - f_L(1-\epsilon)\overline{K}^0\right] \\ &= \frac{1}{2(1+\epsilon)}\left[(f_S+f_L)(1+\epsilon)K^0 + (f_S-f_L)(1-\epsilon)\overline{K}^0\right]\end{aligned}$$

Therefore, the l^+ semi-leptonic decay rate goes as

$$\text{Rate}(l^+) \propto \frac{1}{4}|f_S + f_L|^2 = \frac{1}{4}\left(|f_S|^2 + |f_L|^2 + f_S f_L^* + f_S^* f_L\right) = \frac{1}{4}\left[|f_S|^2 + |f_L|^2 + 2Re(f_S f_L^*)\right]$$

This gives

$$\text{Rate}(l^+) \propto \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right]$$

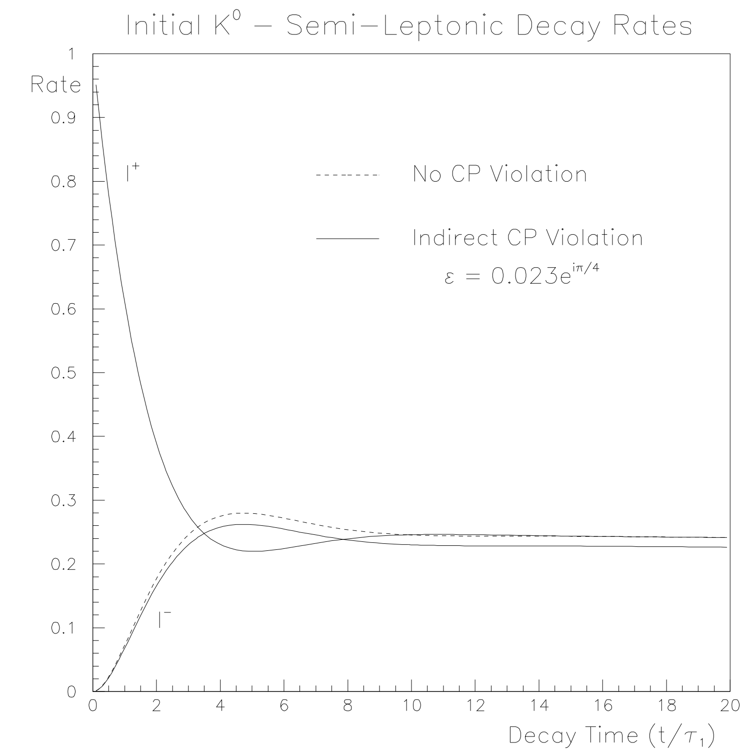
Similarly, the l^- semi-leptonic decay rate goes as

$$\text{Rate}(l^-) \propto \frac{1}{4}\frac{|1-\epsilon|^2}{|1+\epsilon|^2}\left[|f_S|^2 + |f_L|^2 - 2Re(f_S f_L^*)\right]$$

which gives

$$\text{Rate}(l^-) \propto \frac{1}{4}\frac{|1-\epsilon|^2}{|1+\epsilon|^2}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right]$$

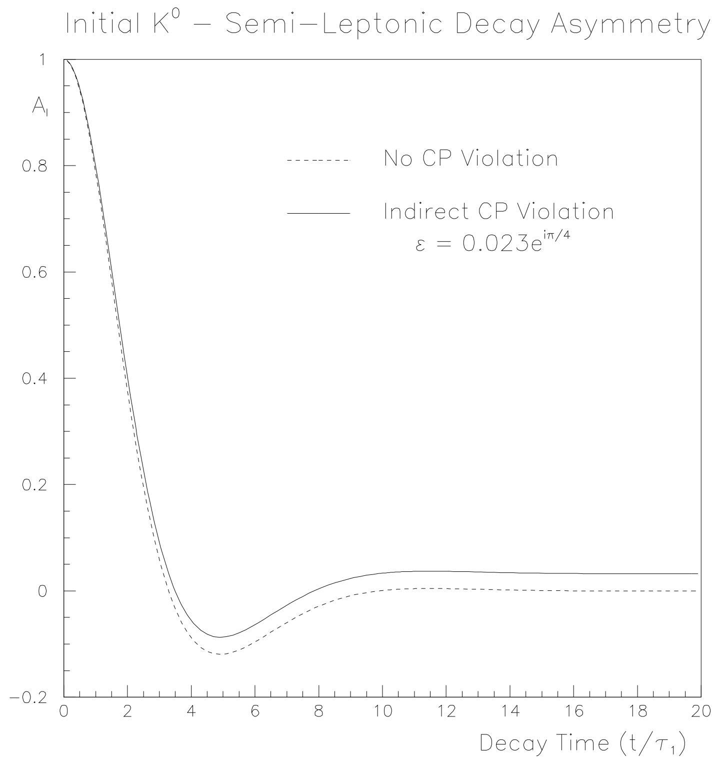
Hence, the positive lepton rate is unaffected by indirect CP violation but the negative lepton rate is reduced by a small factor. These rates are shown below for $\epsilon = 0.023e^{i\pi/4}$, which larger than the actual value by an order of magnitude so as to make the effects visible.

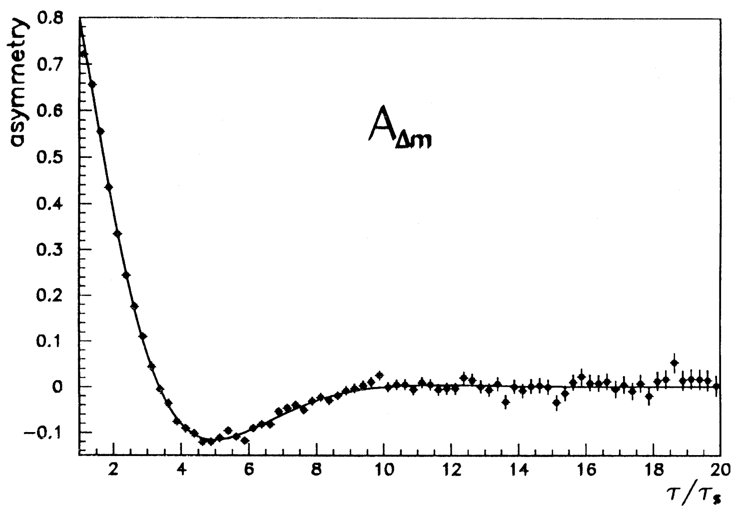


The asymmetry of the rates is

$$\begin{aligned}
A_l &= \frac{\text{Rate}(l^+) - \text{Rate}(l^-)}{\text{Rate}(l^+) + \text{Rate}(l^-)} \\
&= \frac{\left(1 - \frac{|1-\epsilon|^2}{|1+\epsilon|^2}\right)\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + \left(1 + \frac{|1-\epsilon|^2}{|1+\epsilon|^2}\right)\left(2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right)}{\left(1 + \frac{|1-\epsilon|^2}{|1+\epsilon|^2}\right)\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + \left(1 - \frac{|1-\epsilon|^2}{|1+\epsilon|^2}\right)\left(2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right)} \\
&= \frac{\frac{4Re(\epsilon)}{|1+\epsilon|^2}\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + \frac{2(1+|\epsilon|^2)}{|1+\epsilon|^2}\left(2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right)}{\frac{2(1+|\epsilon|^2)}{|1+\epsilon|^2}\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + \frac{4Re(\epsilon)}{|1+\epsilon|^2}\left(2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right)} \\
&= \frac{2Re(\epsilon)\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + (1+|\epsilon|^2)\left(2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right)}{(1+|\epsilon|^2)\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + 2Re(\epsilon)\left(2\cos(\Delta m t)e^{-(\Gamma_S+\Gamma_L)t/2}\right)}
\end{aligned}$$

which is shown below for the same value of ϵ as above, along with experimental data of this quantity.





Note, in the limit of very large times, the $e^{-\Gamma_L t}$ terms dominate and the beam is effectively pure K^0_L. In this limit

$$A_l \approx \frac{2Re(\epsilon)}{(1+|\epsilon|^2)}$$

and hence the K^0_L has a different branching fraction to l^+ as l^-, which is a direct reflection of CP violation.

To find the hadronic decay rates, the pure K^0 state needs to be expressed in terms of K^0_1 and K^0_2, so

$$\psi(t) \quad = \quad \frac{1}{N\sqrt{2}(1+\epsilon)} N \left[f_S \left(K_1^0 + \epsilon K_2^0 \right) + f_L \left(K_2^0 + \epsilon K_1^0 \right) \right]$$

$$= \frac{1}{\sqrt{2}(1+\epsilon)} \left[\left(f_S + \epsilon f_L\right) K_1^0 + \left(f_L + \epsilon f_S\right) K_2^0 \right]$$

Hence, the two pion rate goes as

$$\text{Rate}(2\pi) \propto \frac{1}{2|1+\epsilon|^2} |f_S + \epsilon f_L|^2 = \frac{1}{2|1+\epsilon|^2} \left(|f_S|^2 + |\epsilon|^2 |f_L|^2 + 2Re(\epsilon^* f_S f_L^*) \right)$$

Writing $\epsilon = |\epsilon| e^{i\phi}$, then this is

$$\text{Rate}(2\pi) \propto \frac{1}{2|1+\epsilon|^2} \left[e^{-\Gamma_S t} + |\epsilon|^2 e^{-\Gamma_L t} + 2|\epsilon| \cos(\Delta m t - \phi) e^{-(\Gamma_S + \Gamma_L)t/2} \right]$$

Hence, compared with CP conserved case, the overall rate is reduced by a small amount, there is an oscillation with amplitude $|\epsilon|$ and a long-lived component with size $|\epsilon|^2$. It is this long-lived component which gives the 2π rate which was observed to discover CP violation originally.

For an initial $\overline{K}^0$ beam, the state is

$$\psi(t) = \frac{1}{N\sqrt{2}(1-\epsilon)} \left(f_S(t) K_S^0 - f_L(t) K_L^0 \right)$$

so, in terms of K_1^0 and K_2^0

$$\begin{aligned} \psi(t) &= \frac{1}{N\sqrt{2}(1-\epsilon)} N \left[f_S \left(K_1^0 + \epsilon K_2^0 \right) - f_L \left(K_2^0 + \epsilon K_1^0 \right) \right] \\ &= \frac{1}{\sqrt{2}(1-\epsilon)} \left[\left(f_S - \epsilon f_L\right) K_1^0 - \left(f_L - \epsilon f_S\right) K_2^0 \right] \end{aligned}$$

and so the two pion rate goes as

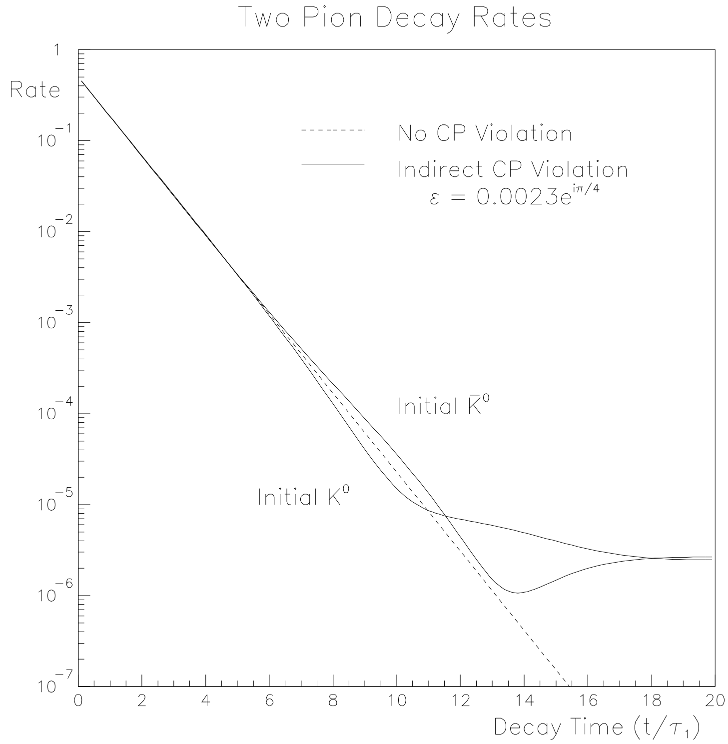
$$\text{Rate}(2\pi) \propto \frac{1}{2|1-\epsilon|^2} |f_S - \epsilon f_L|^2 = \frac{1}{2|1-\epsilon|^2} \left(|f_S|^2 + |\epsilon|^2 |f_L|^2 - 2Re(\epsilon^* f_S f_L^*) \right)$$

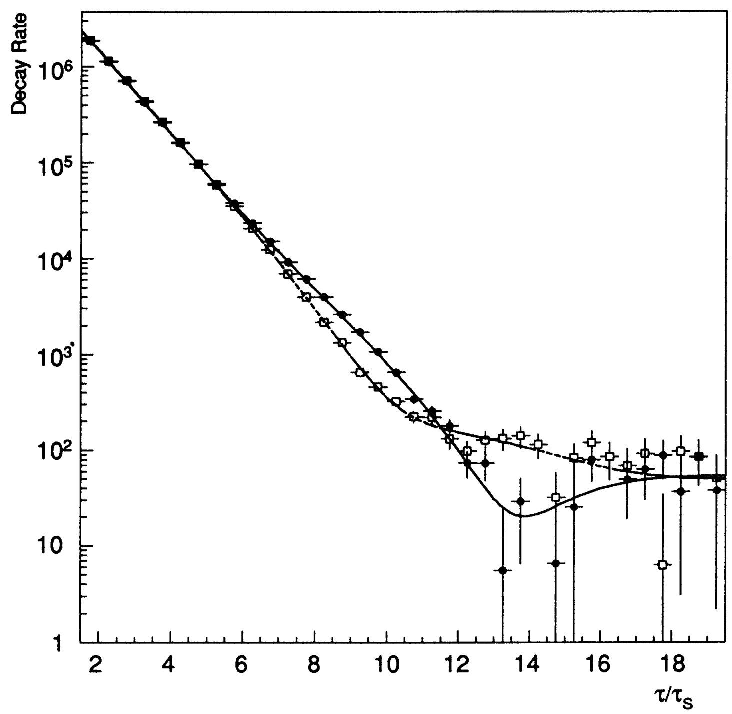
which is

$$\text{Rate}(2\pi) \propto \frac{1}{2|1-\epsilon|^2} \left[e^{-\Gamma_S t} + |\epsilon|^2 e^{-\Gamma_L t} - 2|\epsilon| \cos(\Delta m t - \phi) e^{-(\Gamma_S + \Gamma_L)t/2} \right]$$

Hence, in this case the overall rate is increased by a small amount, there is again an oscillation with amplitude $|\epsilon|$ but with an opposite sign, and again a long-lived component with size $|\epsilon|^2$.

These two are shown below for the measured value of ϵ, along with experimental data.





The asymmetry of these rates is

$$
\begin{aligned}
A_{2\pi} &= \frac{\text{Rate(Initial } K^0 \to 2\pi) - \text{Rate(Initial } \overline{K}^0 \to 2\pi)}{\text{Rate(Initial } K^0 \to 2\pi) + \text{Rate(Initial } \overline{K}^0 \to 2\pi)} \\
&= \frac{\left(\frac{1}{|1+\epsilon|^2} - \frac{1}{|1-\epsilon|^2}\right)\left(e^{-\Gamma_S t} + |\epsilon|^2 e^{-\Gamma_L t}\right) + \left(\frac{1}{|1+\epsilon|^2} + \frac{1}{|1-\epsilon|^2}\right)\left(2|\epsilon|\cos(\Delta m t - \phi)e^{-(\Gamma_S+\Gamma_L)t/2}\right)}{\left(\frac{1}{|1+\epsilon|^2} + \frac{1}{|1-\epsilon|^2}\right)(e^{-\Gamma_S t} + |\epsilon|^2 e^{-\Gamma_L t}) + \left(\frac{1}{|1+\epsilon|^2} - \frac{1}{|1-\epsilon|^2}\right)(2|\epsilon|\cos(\Delta m t - \phi)e^{-(\Gamma_S+\Gamma_L)t/2})}
\end{aligned}
$$

$$= \quad \frac{-2Re(\epsilon)\left(e^{-\Gamma_S t} + |\epsilon|^2 e^{-\Gamma_L t}\right) + (1+|\epsilon|^2)\left(2|\epsilon|\cos(\Delta m t - \phi) e^{-(\Gamma_S+\Gamma_L)t/2}\right)}{(1+|\epsilon|^2)\left(e^{-\Gamma_S t} + |\epsilon|^2 e^{-\Gamma_L t}\right) - 2Re(\epsilon)\left(2|\epsilon|\cos(\Delta m t - \phi) e^{-(\Gamma_S+\Gamma_L)t/2}\right)}$$

which is shown below. Also shown are experimental results, but note this asymmetry definition is inverted compared with the one calculated above.