Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 5

Hand in by 15/03/ for Rapid Feedback on 17/03

1. This question looks at conservation laws in hadron decays.

   Qualitatively explain the significance of the following observations;

   - The partial widths for decays to photons; $\Gamma(\pi^0 \to \gamma\gamma) = 7.84$ eV but $\Gamma(\pi^0 \to \gamma\gamma\gamma)$ has not been seen and has a limit of $< 2\times10^{-7}$ eV. Similarly, $\Gamma(\eta \to \gamma\gamma) = 460$ eV but $\Gamma(\eta \to \gamma\gamma\gamma)$ has not been seen and has a limit of < 0.6 eV and also $\Gamma(\eta' \to \gamma\gamma) = 4300$ eV but $\Gamma(\eta' \to \gamma\gamma\gamma)$ has not been seen and has a limit of < 20 eV.
   - The partial widths for decays to electrons; $\Gamma(\pi^0 \to e^+e^-) = 6 \times 10^{-7}$ eV but $\Gamma(\pi^0 \to e^+e^-e^+e^-) = 2\times10^{-4}$ eV. However, $\Gamma(\rho^0 \to e^+e^-) = 6.8$ keV but $\Gamma(\rho^0 \to e^+e^-e^+e^-)$ has not been seen.
   - The partial widths for decays to hadrons; $\Gamma(\eta \to \pi^+\pi^-\pi^0) = 0.38$ keV and $\Gamma(\eta \to \pi^0\pi^0\pi^0) = 0.27$ keV but $\Gamma(\eta \to \pi^+\pi^-)$ has not been seen and has a limit of < 1.8 eV. Similarly, $\Gamma(\eta' \to \pi^0\pi^0\pi^0) = 0.31$ keV but $\Gamma(\eta' \to \pi^+\pi^-)$ has not been seen. However, $\Gamma(\rho^0 \to \pi^+\pi^-) = \Gamma(\rho^\pm \to \pi^\pm\pi^0) = 150$ MeV, but neither $\Gamma(\rho^0 \to \pi^0\pi^0)$ nor $\Gamma(\rho^0 \to \eta\pi^0)$ have been seen whereas $\Gamma(\rho^0 \to \pi^0\gamma) = 120$ keV and $\Gamma(\rho^0 \to \eta\gamma) = 57$ keV.

2. This question shows in the general case of any operator with only two distinct eigenvalues, projection operators can be formed.

   Consider a operator $\hat{A}$ which has several eigenstates ψ_i with eigenvalues λ_i, so

$$\hat{A}\psi_i = \lambda_i\psi_i$$

   The eigenstates form a complete set, so that any state can in general be decomposed into a sum of the eigenstates

$$\Psi = \sum_i \alpha_i\psi_i$$

   for some coefficients α_i. Show that applying the combination $(\lambda_1 - \hat{A})$ to the general state Ψ removes the eigenstate ψ_1 from this sum. What happens to the coefficients of the other eigenstates during this operation?

   Now consider the case where $\hat{A}$ has only two distinct eigenvalues. Show that the projection operators

$$\hat{P}_1 = \frac{\lambda_2 - \hat{A}}{\lambda_2 - \lambda_1}$$

   and

$$\hat{P}_2 = \frac{\lambda_1 - \hat{A}}{\lambda_1 - \lambda_2}$$

   each remove one of the two eigenvectors from the general state Ψ and leave the coefficient of the other eigenvector unchanged.

   It is convenient to define a new operator from $\hat{A}$

$$\hat{B} = \frac{2\hat{A} - (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)}$$

Show that $\hat{B}$ has the same eigenstates as $\hat{A}$, but with eigenvalues of ± 1, i.e. that

$$\hat{B}\psi_1 = \psi_1, \qquad \hat{B}\psi_2 = -\psi_2$$

By considering the general state

$$\Psi = \alpha_1\psi_1 + \alpha_2\psi_2$$

then show that $\hat{B}^2 = 1$, i.e.

$$\hat{B}^2\Psi = \Psi$$

Show that, in terms of $\hat{B}$, the projection operators are

$$P_1 = \frac{1}{2}(1+\hat{B}), \qquad P_2 = \frac{1}{2}(1-\hat{B})$$

Hence, show that the $P_{1,2}$ have the other required properties for projection operators, namely

$$P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1P_2 = P_2P_1 = 0, \quad P_1 + P_2 = 1$$

3. This question looks at the handedness operator and applies the projection operator results from the previous question.

   The handedness operator is $\gamma^5/2$. Using the relation $\gamma^5\gamma^5 = 1$, show the eigenvalues of handedness are $\pm 1/2$. Taking the handedness operator as $\hat{A}$ in question 1 above, construct the operator $\hat{B}$ and hence find the projection operators P_1 and P_2.

   Show the handedness operator does not commute with the Dirac Hamiltonian. Also, show that the current $\overline{\psi}\gamma^\mu\gamma^5\psi$ is not conserved. Under what condition do both these results change?

   In the standard representation, the handedness operator is

$$\frac{1}{2}\gamma^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

   Show that, for any values of a and b, the states

$$\begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}, \qquad \begin{pmatrix} a \\ b \\ -a \\ -b \end{pmatrix}$$

   are eigenvectors of handedness and find their eigenvalues.

   In the standard representation, the particle solutions of the Dirac equation with momenta along the z axis and spin $\pm 1/2$ along the z axis are

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p/(E+m) \\ 0 \end{pmatrix}, \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -p/(E+m) \end{pmatrix}$$

   These are clearly therefore eigenstates of helicity with eigenvalues $\pm 1/2$ respectively. In which limit are they also eigenstates of handedness and what are their eigenvalues?

In general, any state can be decomposed into right-handed (+1/2) and left-handed (−1/2) eigenstates

$$\Psi = a\psi_R + b\psi_L$$

Write down the explicit form of the projection operators in the standard representation and hence, explicitly find $a\psi_R$ and consequently $|a|^2(\psi_R^\dagger \psi_R)$ for $\Psi = u_1$. With the ψ_R normalisation set to be the same as for the u_i, i.e. $u_i^\dagger u_i = 2E$, then show

$$|a| = \sqrt{\frac{1+\beta}{2}}$$

Similarly show

$$|b| = \sqrt{\frac{1-\beta}{2}}$$

4. This question works through the calculation of the $W^\pm$ decay widths and branching fractions.

   The spin-averaged matrix element of the decay $W^- \to e^-\overline{\nu}_e$ can be calculated from the Feynman diagram to be

   $$\langle |M|^2 \rangle = \frac{g_W^2 M_W^2}{3}$$

   where the electron mass has been neglected. Show that the partial width for this mode is therefore

   $$\Gamma(W^- \to e^-\overline{\nu}_e) = \frac{G_F M_W^3}{6\pi\sqrt{2}}$$

   and evaluate this numerically.

   List the possible decay modes for the $W^\pm$ to leptons or quarks and calculate the partial widths for the other lepton decays and for decays to hadrons, neglecting all fermion masses and higher-order QCD effects. Calculate the total width and hence lifetime of the $W^\pm$ and find the leptonic and hadronic branching fractions. What fractions of the hadronic decays contain charmed and bottom hadrons?

5. Exam question: 2003 question 2.

   The charged pion, with spin zero and mass 139.6 MeV, can decay to an electron, mass 0.511 MeV, or a muon, mass 105.7 MeV, through the decay

   $$\pi^- \to l^- + \overline{\nu}_l,$$

   where l stands for e or μ. These decays have branching fractions of 1.23×10^{-4} and 0.99988 respectively. Any neutrino mass should be neglected in the following.

   (i) Draw a Feynman diagram for this decay.

   (ii) Fermi's Golden Rule gives the partial width Γ_i for a particle of mass m to decay to a mode i to be

   $$\Gamma_i = \frac{|M_i|^2 \rho_i}{2m},$$

   where M_i is the matrix element and ρ_i the Lorentz invariant phase space. Briefly explain the physical significance of the terms in this equation.

(iii) The phase space available for the above pion decays is

$$\rho_l = \frac{1}{8\pi}\frac{m_\pi^2 - m_l^2}{m_\pi^2}.$$

Evaluate the ratio of the magnitudes of the phase space factors for these two decays and comment on your result.

(iv) The first-order matrix element for these decays is

$$|M_l|^2 = 2G_F^2 f_\pi^2 m_l^2 (m_\pi^2 - m_l^2).$$

where G_F is the Fermi constant and f_π is the pion form factor. Calculate an expression for the ratio of the partial widths for these two decays. Evaluate this ratio and comment on your result.

(v) A left-handed state of a fermion has components of both helicity $\pm 1/2$ states, with amplitudes of $\sqrt{(1 \mp \beta)/2}$ respectively, where β is the velocity of the particle. Assuming the weak interactions couple only to left-handed particles (and hence right-handed antiparticles), draw a diagram showing the lepton and antineutrino helicities in these decays.

(vi) From the expression for the matrix element in part (iv) above, the decay rate becomes zero as the lepton mass goes to zero. Explain this observation in terms of your diagram and hence qualitatively explain why the electron decay is heavily suppressed compared with the muon decay.

(vii) The charged kaon has a mass of 493.7 MeV. The branching fraction for the equivalent muon decay $K^- \to \mu^- \overline{\nu}_\mu$ is 0.6351. Briefly explain why this is lower than for the pion case and estimate the branching fraction for the decay $K^- \to e^- \overline{\nu}_e$.

6. CP violation in neutral kaons is dominated by indirect CP violation. This question shows the results of this form of the effect. **This is a long and complex question and is *not* for rapid feedback.**

The $d\overline{s}$ and $s\overline{d}$ quark states are written as K^0 and $\overline{K}^0$ respectively. These are antiparticles of each other and so CP eigenstates can be constructed

$$K_1^0 = \frac{1}{\sqrt{2}}\left(K^0 + \overline{K}^0\right), \qquad K_2^0 = \frac{1}{\sqrt{2}}\left(K^0 - \overline{K}^0\right)$$

where

$$CP(K_1^0) = K_1^0, \qquad CP(K_2^0) = -K_2^0$$

With indirect CP violation, the physical (mass) eigenstates do not correspond exactly to the CP eigenstates but differ by a small amount ϵ

$$K_S^0 = N\left(K_1^0 + \epsilon K_2^0\right), \qquad K_L^0 = N\left(K_2^0 + \epsilon K_1^0\right)$$

where the normalisation factor is

$$N = \frac{1}{\sqrt{1 + |\epsilon|^2}}$$

(Note that because both of these equations contain $+\epsilon$, the states K_S^0 and K_L^0 are not orthogonal; this is related to the fact that the Hamiltonian is not Hermitian.)

Express K_S^0 and K_L^0 in terms of K^0 and $\overline{K}^0$ and then invert the equations to express K^0 and $\overline{K}^0$ in terms of K_S^0 and K_L^0.

The major decays of neutral kaons are semi-leptonic and hadronic. The semi-leptonic decays are

$$K^0 \rightarrow l^+ \, \nu_l \, \pi^-, \qquad \overline{K}^0 \rightarrow l^- \, \overline{\nu}_l \, \pi^+$$

which will have the same partial width in the absence of direct CP violation. The hadronic decays are to two and three pions and in the absence of direct CP violation, these are only

$$K_1^0 \rightarrow 2\pi, \qquad K_2^0 \rightarrow 3\pi$$

but do not have the same partial width.

For a beam of initially pure K^0 particles, find the time dependence of the semi-leptonic rates and hence find a time dependent expression for the asymmetry

$$A_l = \frac{\text{Rate}(l^+) - \text{Rate}(l^-)}{\text{Rate}(l^+) + \text{Rate}(l^-)}$$

How does this compare with the CP conserving case (where $\epsilon = 0$)?

Find the time dependence of the two pion rate for a beam of initially pure K^0 particles. Repeat the calculation of the two pion rate for an initially pure $\overline{K}^0$ beam. Hence, find an expression for

$$A_{2\pi} = \frac{\text{Rate(Initial } K^0 \rightarrow 2\pi) - \text{Rate(Initial } \overline{K}^0 \rightarrow 2\pi)}{\text{Rate(Initial } K^0 \rightarrow 2\pi) + \text{Rate(Initial } \overline{K}^0 \rightarrow 2\pi)}$$

Again, compare these answers with the CP conserving case.