Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 4 Answers

1. The momentum is given by

$$p = \gamma\beta m \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Hence, rearranging

$$m = \frac{p}{\gamma\beta} = \frac{p\sqrt{1-\beta^2}}{\beta}$$

(a) The expected time for a particle to go a distance l is given by

$$\beta = \frac{l}{\langle t \rangle}$$

so

$$\langle t \rangle = \frac{l}{\beta}$$

But, from above

$$(1-\beta^2)p^2 = \beta^2 m^2$$

so

$$1-\beta^2 = \beta^2 \frac{m^2}{p^2}$$

and

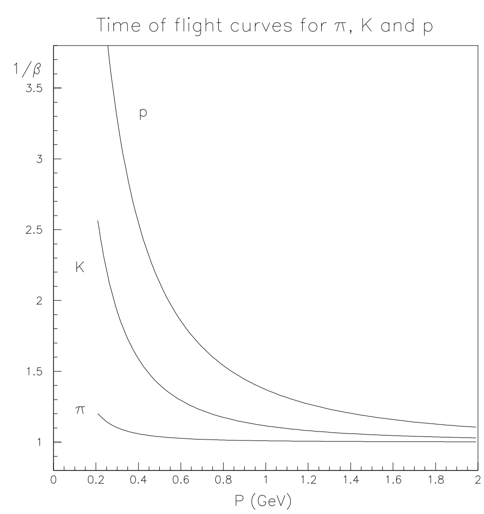
$$\beta^2 \left(1 + \frac{m^2}{p^2}\right) = 1$$

so

$$\frac{1}{\beta} = \sqrt{1 + \frac{m^2}{p^2}}$$

Therefore, the expected time is

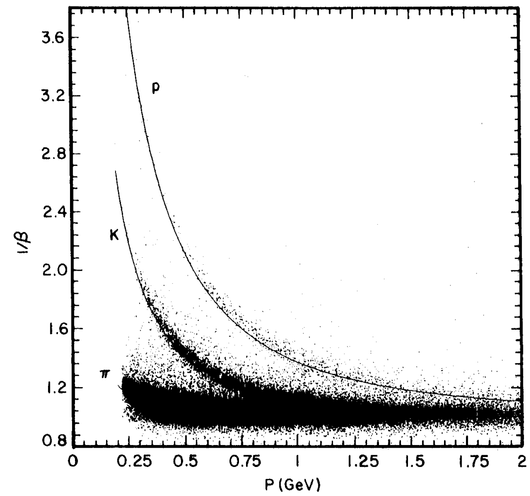
$$\langle t \rangle = \frac{l}{\beta} = l\sqrt{1 + \frac{m^2}{p^2}}$$



For a momentum of 830 MeV and length of 90 cm, then

$$\langle t_\pi \rangle = 3.042 \text{ ns} \qquad \langle t_K \rangle = 3.491 \text{ ns}$$

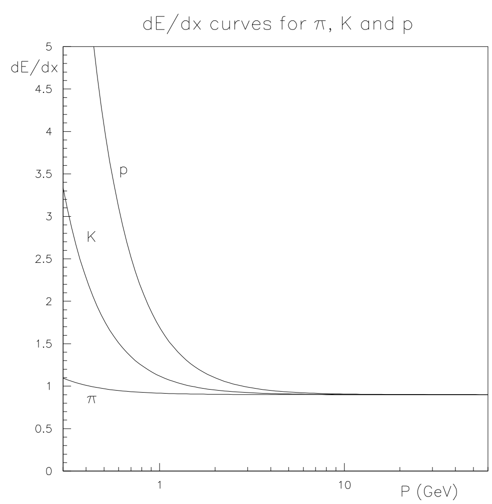
which are different by 449 ps, i.e. approximately $3\sigma_t$. The plot of the CLEO time-of-flight data is shown below



(b) Using the result for $1/\beta$ from above, then

$$-\left\langle \frac{dE}{dx} \right\rangle = \frac{A}{\beta^2} = A\left(1 + \frac{m^2}{p^2}\right)$$

directly.



For pions and kaons to be distinguished, then

$$A\left(1 + \frac{m_K^2}{p^2}\right) - A\left(1 + \frac{m_\pi^2}{p^2}\right) > 3\sigma_E$$

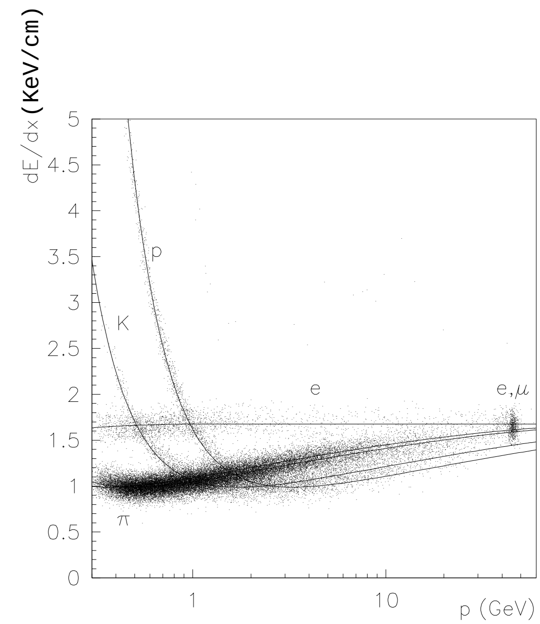
so

$$A\frac{m_K^2 - m_\pi^2}{p^2} > 3\sigma_E$$

and hence

$$p < \sqrt{A \frac{m_K^2 - m_\pi^2}{3\sigma_E}} = 917 \text{ MeV}$$

The plot for the Aleph dE/dx is shown below



Ionisation: The electric field of the particle exerts a force on the atomic electrons and hence ionises the atoms. The energy required for this comes from the kinetic energy of the charged particle. For a given trajectory, the total momentum transfer (the impulse) from the charged particle to the atomic electron depends on the time for which is the charged particle is close to the electron, i.e. inversely depends on the charged particle speed, β. The electrons do not acquire enough energy to become relativistic, so their energy gain goes as the momentum transfer squared; this is therefore $1/\beta^2$. Hence, the energy lost from the charged particle would be expected to be proportional to $1/\beta^2$.

(c) The requirement for Cherenkov radiation is that

$$\beta > \frac{1}{n}$$

which is equivalent to requiring that $\cos\theta$ is less than 1. Hence

$$\frac{1}{\beta} = \sqrt{1 + \frac{m^2}{p^2}} < n$$

so

$$\frac{m^2}{p^2} < n^2 - 1$$

and so

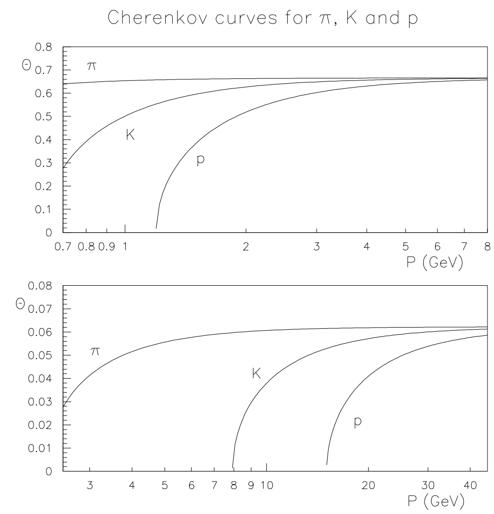
$$p > \frac{m}{\sqrt{n^2 - 1}}$$

From above, the angle is given by

$$\theta = \cos^{-1}\left(\frac{1}{\beta n}\right)$$

so the expected value is

$$\langle\theta\rangle = \cos^{-1}\left(\frac{1}{n}\sqrt{1+\frac{m^2}{p^2}}\right)$$

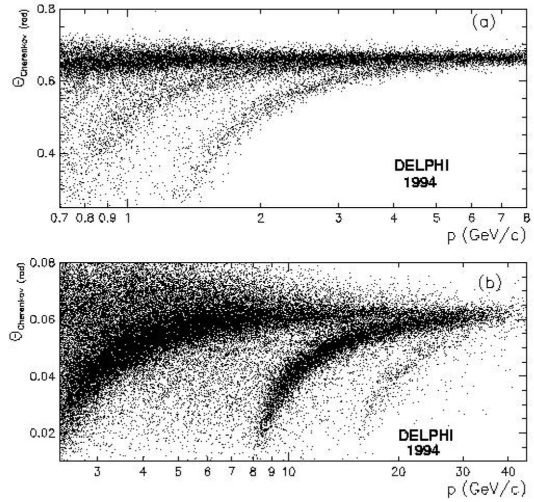


The maximum Cherenkov angle is for particles travelling very close to the speed of light, when $\beta = 1$ and so

$$\cos\theta_{max} = \frac{1}{n}$$

and this gives $\theta_{max} = 38^\circ = 0.666$ rad for the liquid and $\theta_{max} = 3.6^\circ = 0.062$ rad for the gas.

The threshold momenta for the liquid are $p_\pi = 0.178$ GeV and $p_K = 0.628$ GeV and for the gas are $p_\pi = 2.24$ GeV and $p_K = 7.92$ GeV. Therefore, below the lowest threshold of ~ 180 MeV, there are no signals and pions and kaons cannot be distinguished. Between 180 MeV and 630 MeV, the pions give signals in the liquid, but the kaons do not, so they can be separated using the system in “veto” mode, where the presence or absence of a signal distinguishes the particles. Between 630 MeV and 2.2 GeV, both give signals in the liquid but not the gas. They are hardest to distinguish at 2.2 GeV, where $\theta_\pi = 0.663$ rad and $\theta_K = 0.634$ rad which are different by 0.029 rad, while $3\sigma_\theta = 0.012$ rad. Between 2.2 GeV and 7.9 GeV, the pions give signals in the gas also, so again, they can be distinguished using veto mode in the gas radiator. Above 7.9 GeV, both give signals in both the liquid and the gas; in this case, the gas can distinguish them. At 13 GeV, then $\theta_\pi = 0.061$ rad and $\theta_K = 0.049$ rad which are different by 0.012 rad, which is $3\sigma_\theta$. Hence, pions and kaons can be distinguished over the whole range from 180 MeV to 13 GeV using a combination of signals and vetos. Plots for the Delphi Cherenkov system are shown below.



2. Let a_n, b_n and c_n be the average number of photons, electrons and positrons respectively entering layer n. Within this layer, on average pa_n photons will convert into e^+e^- pairs. Also on average pb_n electrons and pc_n positrons will bremsstrahlung. Therefore, the average numbers emerging from layer n will be

$$\begin{aligned} a_{n+1} &= (1-p)a_n + pb_n + pc_n \\ b_{n+1} &= b_n + pa_n \\ c_{n+1} &= c_n + pa_n \end{aligned}$$

Hence

$$a_{n+1} + b_{n+1} + c_{n+1} = (1-p)a_n + pb_n + pc_n + b_n + pa_n + c_n + pa_n = (1+p)(a_n + b_n + c_n)$$

and so the number per layer increases on average by $1+p$. The fraction of particles which are photons is

$$f_n = \frac{a_n}{a_n + b_n + c_n}$$

so

$$\begin{aligned} f_{n+1} &= \frac{(1-p)a_n + pb_n + pc_n}{(1+p)(a_n + b_n + c_n)} \\ &= \frac{1-p}{1+p}\frac{a_n}{a_n + b_n + c_n} + \frac{p}{1+p}\frac{b_n + c_n}{a_n + b_n + c_n} \\ &= \frac{1-p}{1+p} f_n + \frac{p}{1+p}(1 - f_n) \\ &= \frac{1-2p}{1+p} f_n + \frac{p}{1+p} \end{aligned}$$

Assuming f_n tends to a constant for large n, then $f_{n+1} = f_n$ and

$$(1+p)f_n = (1-2p)f_n + p$$

so

$$3pf_n = p \qquad \text{so} \qquad f_n = \frac{1}{3}$$

Electrons and positrons only occur because of photon conversions and are not absorbed in this model, so there must be an equal number of these. Hence, the fraction of electrons (and also positrons) is also 1/3. The only difference for a shower started by an electron would be that there would be one more electron than positron. However, for large n, this difference would be negligible compared with the total number of particles, so the fractions would again be almost equal for all three types of particles.

The average number of particles after each layer is $1+p$ more than the previous layer, so the average total number after layer n is $(1+p)^n$. If the energy is evenly divided, then the average energy of these is $E_0/(1+p)^n$. Hence, the shower will stop on average when

$$\frac{E_0}{(1+p)^n} = E_c$$

Rearranging

$$(1+p)^n = \frac{E_0}{E_c}$$

so

$$n \ln(1+p) = \ln\left(\frac{E_0}{E_c}\right)$$

and

$$n = \frac{\ln(E_0/E_c)}{\ln(1+p)}$$

For a lead sheet of thickness 2mm and a radiation length $X_0 = 5.6$ mm, then the probability of an interaction is

$$p \sim 1 - e^{-2/5.6} \sim 0.3$$

neglecting the difference of 7/9 between electron and photon interaction lengths. Hence, for $E_0 = 104$ GeV, then

$$n = \frac{\ln(E_0/E_c)}{\ln(1+p)} = 36.6$$

so the shower stops around layer 37 on average. This is a total amount of lead of 74 mm which is 13.2 radiation lengths. The total calorimeter is 57 sheets or 114 mm which is 20.4 radiation lengths and helps catch deeper showers from statistical fluctuations around the average. The chance of a photon passing straight through is $e^{-20.4} \sim 10^{-9}$.

3. (iv) The straight line trajectory can be taken as

$$y = mx + y_0$$

and since this passes through (x_1, y_1) and (x_2, y_2), then

$$y_1 = mx_1 + y_0, \qquad y_2 = mx_2 + y_0$$

Hence,

$$y_2 - y_1 = m(x_2 - x_1)$$

so

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting back into the above equation

$$y_1 = mx_1 + y_0 = \frac{y_2 - y_1}{x_2 - x_1} x_1 + y_0$$

so

$$y_0 = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 = \frac{y_1(x_2 - x_1) - x_1(y_2 - y_1)}{x_2 - x_1} = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

The partial derivatives are

$$\frac{\partial y_0}{\partial y_1} = \frac{x_2}{x_2 - x_1}, \qquad \frac{\partial y_0}{\partial y_2} = -\frac{x_1}{x_2 - x_1}$$

so σ_0, the error on y_0, is then given by

$$\sigma_0^2 = \left(\frac{\partial y_0}{\partial y_1}\right)^2 \sigma^2 + \left(\frac{\partial y_0}{\partial y_2}\right)^2 \sigma^2 = \frac{x_2^2 + x_1^2}{(x_2 - x_1)^2} \sigma^2$$

so

$$\sigma_0 = \frac{\sqrt{x_2^2 + x_1^2}}{x_2 - x_1} \sigma$$

(v) For Aleph, $x_1 = 6.3$ cm, $x_2 = 10.8$ cm and $\sigma = 12$ μm, so

$$\sigma_0 = 3.3 \times 10^{-5} \text{ m} = 33 \ \mu\text{m}$$

A typical B meson decay impact parameter is $c\tau_B \approx 480$ μm, which is the track quantity estimated above. (The decay length is longer by a factor $\gamma\beta$ and at Aleph, this factor is on average ~ 7, but the calculation here is for the impact parameter error, not the vertex error.) In either case, the extrapolation error is much smaller. Hence, silicon vertex detectors can be used to measure lifetime effects from B meson decays.

The resolution might not be as good as the above in practice because of multiple scattering. This is the interaction of charged particles with the nuclear electric field, which tends to bend the particle trajectory and hence prevent an accurate extrapolation. This is most important for low momentum particles.

4. (i) The pion states have quark compositions: π^+ is $u\overline{d}$, π^- is $d\overline{u}$ and π^0 is a mixture of $u\overline{u}$ and $d\overline{d}$, specifically $(u\overline{u} - d\overline{d})/\sqrt{2}$. The kaon states are: K^+ is $u\overline{s}$, K^- is $s\overline{u}$, K^0 is $d\overline{s}$ and $\overline{K}^0$ is $s\overline{d}$.

There is no $\overline{\pi}^0$ as this particle is its own antiparticle, as is obvious from its quark composition.

(ii) The particle-antiparticle masses (i.e. π^+ and π^-, K^+ and K^-, K^0 and $\overline{K}^0$) are exactly equal if CPT is conserved. The near equality of the $\pi^\pm$ and π^0 (and also the $K^\pm$ and neutral K states) is due to both colour force universality (so the forces between the quarks are equal except for their electromagnetic forces) and also the near equality of the u and d quark masses.

(iii) Fermions and antifermions have opposite relative intrinsic parity, so a fermion-antifermion bound state has $P = P_f P_{\overline{f}} P_L = (-1)^{L+1}$. Since the parity of these states is -1, then they must have even orbital angular momentum; they actually have orbital angular momentum of zero. Hence, the total spin of the state is

$$\boldsymbol{J} = \boldsymbol{s}_i + \boldsymbol{s}_j$$

Therefore

$$\boldsymbol{J}^2 = \boldsymbol{s}_i^2 + \boldsymbol{s}_j^2 + 2\boldsymbol{s}_i.\boldsymbol{s}_j$$

Hence, the expectation value of the spin term

$$\langle \boldsymbol{s}_i . \boldsymbol{s}_j \rangle = \frac{1}{2} \left[\langle \boldsymbol{J}^2 \rangle - \langle \boldsymbol{s}_i^2 \rangle - \langle \boldsymbol{s}_j^2 \rangle \right]$$

which becomes, using the eigenvalue for the square of any angular momentum,

$$\langle \boldsymbol{s}_i . \boldsymbol{s}_j \rangle = \frac{1}{2} \left[J(J+1) - s_i(s_i+1) - s_j(s_j+1) \right]$$

so

$$\langle \boldsymbol{s}_i . \boldsymbol{s}_j \rangle = \frac{1}{2} \left[J(J+1) - \frac{3}{4} - \frac{3}{4} \right] = \frac{1}{2} J(J+1) - \frac{3}{4}$$

(iv) For the ground states, $J = 0$, so this term is $-3/4$, while for the excited states, $J = 1$ and the term is $+1/4$. With $m_u \approx m_d \equiv m_q$, then the contributions to the masses of the states are

$$m_\pi = 2m_q - \frac{3B}{4m_q^2}$$

$$m_K = m_q + m_s - \frac{3B}{4m_q m_s}$$

$$m_\rho = 2m_q + \frac{B}{4m_q^2}$$

$$m_{K^*} = m_q + m_s + \frac{B}{4m_q m_s}$$

This gives four equations with three unknowns and so allows a check for consistency. Adding and subtracting

$$3m_\rho + m_\pi = 8m_q, \qquad m_\rho - m_\pi = \frac{B}{m_q^2}$$

$$3m_{K^*} + m_K = 4m_q + 4m_s, \qquad m_{K^*} - m_K = \frac{B}{m_q m_s}$$

Hence, from the first three in turn, $m_q = 305$ MeV, $B = 5.86 \times 10^7$ MeV^3 and $m_s = 488$ MeV. The final equation evaluates on the left to 396 MeV and on the right to 394 MeV. Hence, the agreement is very good and is at the O(MeV) level.

(v) For the K^+ with $u\bar{s}$, both particles are positively charged, so the repulsive Coulomb potential contributes to the mass

$$\Delta m_C = \frac{(e/3)(2e/3)}{4\pi\epsilon_0} \frac{1}{\langle r \rangle} = \frac{2e^2}{36\pi\epsilon_0} \frac{1}{\langle r \rangle}$$

Using $\alpha = e^2/4\pi\epsilon_0 = 1/137$ and $1/\langle r \rangle = 3.33$ fm^{-1}, which is 660 MeV, then

$$\Delta m_C = \frac{2\alpha}{9} \frac{1}{\langle r \rangle} = 1.07 \text{ MeV}$$

For a K^0 with $d\bar{s}$ pair, the particles have opposite charge, so this term would be

$$\Delta m_C = -\frac{\alpha}{9} \frac{1}{\langle r \rangle} = -0.53 \text{ MeV}$$

so the total difference between the K^+ and K^0 due to the Coulomb potential would be 1.6 MeV, with the K^+ heavier than the K^0. Overall, the K^0 is heavier than the K^+ by 4 MeV, so the difference in mass between the u and d quark would be estimated to be $m_d - m_u = 5.6$ MeV.

5. For the first two quarks, the $S = 1$ and $S = 0$ states are given by

$$\begin{aligned}
|1, +1\rangle &= \uparrow_1\uparrow_2 \\
|1, \;\;\, 0\rangle &= \sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) \\
|1, -1\rangle &= \downarrow_1\downarrow_2 \\
|0, \;\;\, 0\rangle &= \sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)
\end{aligned}$$

Note the $S = 1$ states are symmetric under interchange while the $S = 0$ state is antisymmetric.

Adding the third quark to the $S = 0$ state gives

$$\begin{aligned}
|1/2, +1/2\rangle &= |0, \;\;\, 0\rangle \uparrow_3 = \sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\uparrow_3 - \downarrow_1\uparrow_2\uparrow_3) \\
|1/2, -1/2\rangle &= |0, \;\;\, 0\rangle \downarrow_3 = \sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\downarrow_3 - \downarrow_1\uparrow_2\downarrow_3)
\end{aligned}$$

Adding the third quark to the $S = 1$ states will give a different set of $S = 1/2$ and also $S = 3/2$. For this set of $S = 1/2$, the states are

$$\begin{aligned}
|1/2, +1/2\rangle &= \sqrt{\frac{2}{3}}|1, +1\rangle \downarrow_3 - \sqrt{\frac{1}{3}}|1, \;\;\, 0\rangle \uparrow_3 \\
&= \sqrt{\frac{2}{3}} \uparrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) \uparrow_3 \\
&= \sqrt{\frac{2}{3}} \uparrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{1}{6}} \uparrow_1\downarrow_2\uparrow_3 - \sqrt{\frac{1}{6}} \downarrow_1\uparrow_2\uparrow_3
\end{aligned}$$

and

$$\begin{aligned}
|1/2, -1/2\rangle &= \sqrt{\frac{1}{3}}|1, \;\;\, 0\rangle \downarrow_3 - \sqrt{\frac{2}{3}}|1, -1\rangle \uparrow_3 \\
&= \sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) \downarrow_3 - \sqrt{\frac{2}{3}} \downarrow_1\downarrow_2\uparrow_3 \\
&= \sqrt{\frac{1}{6}} \uparrow_1\downarrow_2\downarrow_3 + \sqrt{\frac{1}{6}} \downarrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{2}{3}} \downarrow_1\downarrow_2\uparrow_3
\end{aligned}$$

For the $S = 3/2$ states then

$$\begin{aligned}
|3/2, +3/2\rangle &= |1, +1\rangle \uparrow_3 = \uparrow_1\uparrow_2\uparrow_3 \\
|3/2, +1/2\rangle &= \sqrt{\frac{1}{3}}|1, +1\rangle \downarrow_3 + \sqrt{\frac{2}{3}}|1, \;\;\, 0\rangle \uparrow_3 \\
&= \sqrt{\frac{1}{3}} \uparrow_1\uparrow_2\downarrow_3 + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) \uparrow_3 \\
&= \sqrt{\frac{1}{3}}\,(\uparrow_1\uparrow_2\downarrow_3 + \uparrow_1\downarrow_2\uparrow_3 + \downarrow_1\uparrow_2\uparrow_3) \\
|3/2, -1/2\rangle &= \sqrt{\frac{2}{3}}|1, \;\;\, 0\rangle \downarrow_3 + \sqrt{\frac{1}{3}}|1, -1\rangle \uparrow_3 \\
&= \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) \downarrow_3 + \sqrt{\frac{1}{3}} \downarrow_1\downarrow_2\uparrow_3 \\
&= \sqrt{\frac{1}{3}}\,(\uparrow_1\downarrow_2\downarrow_3 + \downarrow_1\uparrow_2\downarrow_3 + \downarrow_1\downarrow_2\uparrow_3) \\
|3/2, -3/2\rangle &= |1, -1\rangle \downarrow_3 = \downarrow_1\downarrow_2\downarrow_3
\end{aligned}$$

which completes the eight possible states.

Under interchange of 1 and 2, then the first $S = 1/2$ states becomes

$$\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\uparrow_3 - \downarrow_1\uparrow_2\uparrow_3) \rightarrow \sqrt{\frac{1}{2}}\,(\downarrow_1\uparrow_2\uparrow_3 - \uparrow_1\downarrow_2\uparrow_3) = -\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\uparrow_3 - \downarrow_1\uparrow_2\uparrow_3)$$

and

$$\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\downarrow_3 - \downarrow_1\uparrow_2\downarrow_3) \rightarrow \sqrt{\frac{1}{2}}\,(\downarrow_1\uparrow_2\downarrow_3 - \uparrow_1\downarrow_2\downarrow_3) = -\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\downarrow_3 - \downarrow_1\uparrow_2\downarrow_3)$$

and so both are antisymmetric under $1 \leftrightarrow 2$ interchange. This would be expected as the original $S = 0$ state is antisymmetric to this exchange. However, under 1 and 3 interchange

$$\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\uparrow_3 - \downarrow_1\uparrow_2\uparrow_3) \rightarrow \sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\uparrow_3 - \uparrow_1\uparrow_2\downarrow_3)$$

and

$$\sqrt{\frac{1}{2}}\,(\uparrow_1\downarrow_2\downarrow_3 - \downarrow_1\uparrow_2\downarrow_3) \rightarrow \sqrt{\frac{1}{2}}\,(\downarrow_1\downarrow_2\uparrow_3 - \downarrow_1\uparrow_2\downarrow_3)$$

and so have no definite symmetry. The same is true for 2 and 3 interchange.

For the second set of $S = 1/2$ states, then $1 \leftrightarrow 2$ gives

$$\begin{aligned}\sqrt{\frac{2}{3}}\uparrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\uparrow_3 - \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\uparrow_3 \rightarrow & \sqrt{\frac{2}{3}}\uparrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\uparrow_3 - \sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\uparrow_3 \\ = & \sqrt{\frac{2}{3}}\uparrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\uparrow_3 - \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\uparrow_3\end{aligned}$$

and

$$\begin{aligned}\sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\downarrow_3 + \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{2}{3}}\downarrow_1\downarrow_2\uparrow_3 \rightarrow & \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\downarrow_3 + \sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\downarrow_3 - \sqrt{\frac{2}{3}}\downarrow_1\downarrow_2\uparrow_3 \\ = & \sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\downarrow_3 + \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{2}{3}}\downarrow_1\downarrow_2\uparrow_3\end{aligned}$$

which are therefore both symmetric. Again, this would be expected as the original $S = 1$ state is symmetric to this exchange. For $1 \leftrightarrow 3$

$$\sqrt{\frac{2}{3}}\uparrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\uparrow_3 - \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\uparrow_3 \rightarrow \sqrt{\frac{2}{3}}\downarrow_1\uparrow_2\uparrow_3 - \sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\uparrow_3 - \sqrt{\frac{1}{6}}\uparrow_1\uparrow_2\downarrow_3$$

and

$$\sqrt{\frac{1}{6}}\uparrow_1\downarrow_2\downarrow_3 + \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{2}{3}}\downarrow_1\downarrow_2\uparrow_3 \rightarrow \sqrt{\frac{1}{6}}\downarrow_1\downarrow_2\uparrow_3 + \sqrt{\frac{1}{6}}\downarrow_1\uparrow_2\downarrow_3 - \sqrt{\frac{2}{3}}\uparrow_1\downarrow_2\downarrow_3$$

and so do not have a definite symmetry; the same is true for $2 \leftrightarrow 3$.

The final states are the $S = 3/2$. The $S_z = \pm 3/2$ are clearly symmetric under interchange of any pair

$$\uparrow_1\uparrow_2\uparrow_3, \qquad \downarrow_1\downarrow_2\downarrow_3$$

and for the other two, for $1 \leftrightarrow 2$

$$\begin{aligned}\sqrt{\frac{1}{3}}\,(\uparrow_1\uparrow_2\downarrow_3 + \uparrow_1\downarrow_2\uparrow_3 + \downarrow_1\uparrow_2\uparrow_3) \rightarrow & \sqrt{\frac{1}{3}}\,(\uparrow_1\uparrow_2\downarrow_3 + \downarrow_1\uparrow_2\uparrow_3 + \uparrow_1\downarrow_2\uparrow_3) \\ = & \sqrt{\frac{1}{3}}\,(\uparrow_1\uparrow_2\downarrow_3 + \uparrow_1\downarrow_2\uparrow_3 + \downarrow_1\uparrow_2\uparrow_3)\end{aligned}$$

and

$$\sqrt{\frac{1}{3}}(\uparrow_1\downarrow_2\downarrow_3 + \downarrow_1\uparrow_2\downarrow_3 + \downarrow_1\downarrow_2\uparrow_3) \to \sqrt{\frac{1}{3}}(\downarrow_1\uparrow_2\downarrow_3 + \uparrow_1\downarrow_2\downarrow_3 + \downarrow_1\downarrow_2\uparrow_3)$$
$$= \sqrt{\frac{1}{3}}(\uparrow_1\downarrow_2\downarrow_3 + \downarrow_1\uparrow_2\downarrow_3 + \downarrow_1\downarrow_2\uparrow_3)$$

For $1 \leftrightarrow 3$ interchange, then

$$\sqrt{\frac{1}{3}}(\uparrow_1\uparrow_2\downarrow_3 + \uparrow_1\downarrow_2\uparrow_3 + \downarrow_1\uparrow_2\uparrow_3) \to \sqrt{\frac{1}{3}}(\downarrow_1\uparrow_2\uparrow_3 + \uparrow_1\downarrow_2\uparrow_3 + \uparrow_1\uparrow_2\downarrow_3)$$
$$= \sqrt{\frac{1}{3}}(\uparrow_1\uparrow_2\downarrow_3 + \uparrow_1\downarrow_2\uparrow_3 + \downarrow_1\uparrow_2\uparrow_3)$$

and

$$\sqrt{\frac{1}{3}}(\uparrow_1\downarrow_2\downarrow_3 + \downarrow_1\uparrow_2\downarrow_3 + \downarrow_1\downarrow_2\uparrow_3) \to \sqrt{\frac{1}{3}}(\downarrow_1\downarrow_2\uparrow_3 + \downarrow_1\uparrow_2\downarrow_3 + \uparrow_1\downarrow_2\downarrow_3)$$
$$= \sqrt{\frac{1}{3}}(\uparrow_1\downarrow_2\downarrow_3 + \downarrow_1\uparrow_2\downarrow_3 + \downarrow_1\downarrow_2\uparrow_3)$$

and so is also symmetric. The same is again true for $2 \leftrightarrow 3$.

In summary, the first set of $S = 1/2$ states are antisymmetric under $1 \leftrightarrow 2$ interchange but have no definite symmetry for $1 \leftrightarrow 3$ or $2 \leftrightarrow 3$. The second set of $S = 1/2$ states are symmetric under $1 \leftrightarrow 2$ interchange but have no definite symmetry for $1 \leftrightarrow 3$ or $2 \leftrightarrow 3$. The set of $S = 3/2$ states are symmetric under any of $1 \leftrightarrow 2$, $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$. Note, all the different S_z states corresponding to a given S state have the same symmetry, as would be expected since the symmetry should not change just from performing a rotation of the coordinate axes.

In the absence of colour, then the Pauli exclusion principle would say any of the states is possible for uds, as there are no identical particles in that case. However, for uud, then the first set of $S = 1/2$ states would be possible with the uu pair as 1 and 2 but neither of the others states would be allowed. There are no completely antisymmetric states under all three interchanges, so no uuu state would be allowed at all. Hence, there would be seven baryons corresponding to the first $S = 1/2$ state with all the quark combinations excluding uuu, ddd and sss. There would also be one uds baryon in each of the other $S = 1/2$ and $S = 3/2$ states.

The QCD colourless combination under $1 \leftrightarrow 2$ exchange is

$$\sqrt{\frac{1}{6}}(r_1b_2g_3 - r_1g_2b_3 + g_1r_2b_3 - g_1b_2r_3 + b_1g_2r_3 - b_1r_2g_3)$$
$$\to \sqrt{\frac{1}{6}}(b_1r_2g_3 - g_1r_2b_3 + r_1g_2b_3 - b_1g_2r_3 + g_1b_2r_3 - r_1b_2g_3)$$
$$= -\sqrt{\frac{1}{6}}(r_1b_2g_3 - r_1g_2b_3 + g_1r_2b_3 - g_1b_2r_3 + b_1g_2r_3 - b_1r_2g_3)$$

and similarly for the other two and so is completely antisymmetric. Hence, to satisfy the Pauli exclusion principle, the spin states have to now be symmetric under interchange, so that overall the state is antisymmetric. This means for uud, then either the second $S = 1/2$ states, with uu being 1 and 2, or the $S = 3/2$ states are allowed, whereas the first $S = 1/2$ states are now forbidden. For uuu, the $S = 3/2$ states are allowed. Hence, there would be just one (uds) baryon in the first $S = 1/2$ state, seven combinations (all except uuu, ddd and sss) in the second $S = 1/2$ state and all ten possible combinations in the $S = 3/2$ state, as observed.