Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 3 Answers

1. From the given Lagrangian density, differentiating with respect to ϕ^* gives

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2\phi, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi$$

so

$$\partial_\mu \partial^\mu \phi = -m^2 \phi \qquad \text{or} \qquad \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

which is the Klein-Gordon equation. Note, the complex conjugate of this is simply

$$\partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0$$

so ϕ^* also obeys the Klein-Gordon equation, as can also be deduced directly using the Euler-Lagrange equations for ϕ.

For the transformation $\phi \to \phi e^{-i\alpha}$ with constant α, then the Lagrangian density is clearly unchanged so there must be a conserved current. Using

$$\frac{\partial \phi}{\partial \alpha} = -i\phi, \qquad \frac{\partial \phi^*}{\partial \alpha} = i\phi^*$$

then the conserved quantity in the equation given is

$$\frac{\partial L}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial \alpha} + \frac{\partial L}{\partial(\partial_\mu \phi^*)} \frac{\partial \phi^*}{\partial \alpha} = (\partial^\mu \phi^*)(-i\phi) + (i\phi^*)(\partial^\mu \phi)$$

which is

$$J^\mu = i\left[\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi\right]$$

This is the probability current density (and hence charge density when multiplied by q) for spin 0 particles.

Taking derivatives and using the Klein-Gordon equation for both ϕ and ϕ^*

$$\begin{aligned}
\partial_\mu J^\mu &= i\left[(\partial_\mu \phi^*)(\partial^\mu \phi) + \phi^*(\partial_\mu \partial^\mu \phi) - (\partial_\mu \partial^\mu \phi^*)\phi - (\partial^\mu \phi^*)(\partial_\mu \phi)\right] \\
&= i\left[\phi^*(-m^2\phi) - (-m^2\phi^*)\phi\right] = 0
\end{aligned}$$

2. (i) The first and second terms in the Lagrangian are the kinetic and mass terms for a free electron; they together give the free electron Lagrangian. The third term is for a free photon field, i.e. with no charges. The fourth term is the interaction term between the electron and photon.

   (ii) For the case $q \equiv \overline{\psi}$, then

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = \frac{i}{2}\gamma^\mu \partial_\mu \psi - m\psi - eA_\mu \gamma^\mu \psi$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = -\frac{i}{2}\gamma^\mu \psi$$

Hence, the Euler-Lagrange equations yield

$$\frac{i}{2}\gamma^\mu \partial_\mu \psi - m\psi - eA_\mu \gamma^\mu \psi = \partial_\mu\left(-\frac{i}{2}\gamma^\mu \psi\right) = -\frac{i}{2}\gamma^\mu \partial_\mu \psi$$

Rearranging gives

$$i\gamma^\mu \partial_\mu \psi - eA_\mu \gamma^\mu \psi = m\psi$$

or

$$i\gamma^\mu \left(\partial_\mu + ieA_\mu\right) \psi = m\psi$$

which is the Dirac equation in an electromagnetic field. The Hermitian conjugate of this equation is

$$-i\left(\partial_\mu - ieA_\mu\right) \psi^\dagger \gamma^{\mu\dagger} = m\psi^\dagger$$

Hence, using $\gamma^0\gamma^0 = 1$

$$-i\left(\partial_\mu - ieA_\mu\right) \psi^\dagger \gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 = m\psi^\dagger\gamma^0$$

and using $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ then

$$-i\left(\partial_\mu - ieA_\mu\right) \overline{\psi}\gamma^\mu = m\overline{\psi}$$

Consider

$$\partial_\mu(e\overline{\psi}\gamma^\mu\psi) = e(\partial_\mu\overline{\psi})\gamma^\mu\psi + e\overline{\psi}\gamma^\mu(\partial_\mu\psi)$$

From above

$$\gamma^\mu(\partial_\mu\psi) = -ieA_\mu\gamma^\mu\psi - im\psi \qquad (\partial_\mu\overline{\psi})\gamma^\mu = ieA_\mu\overline{\psi}\gamma^\mu + im\overline{\psi}$$

so

$$\partial_\mu(e\overline{\psi}\gamma^\mu\psi) = e\left(ieA_\mu\overline{\psi}\gamma^\mu + im\overline{\psi}\right)\psi + e\overline{\psi}\left(-ieA_\mu\gamma^\mu\psi - im\psi\right) = 0$$

The current $\overline{\psi}\gamma^\mu\psi$ is the Dirac probability current and with the factor e represents the conservation of charge.

(iii) Under the transformation

$$\psi' = \psi e^{-ie\Lambda}$$

then

$$\begin{aligned}
\partial_\mu\psi' &= (\partial_\mu\psi)e^{-ie\Lambda} + \psi(\partial_\mu e^{-ie\Lambda}) \\
&= (\partial_\mu\psi)e^{-ie\Lambda} - ie(\partial_\mu\Lambda)\psi e^{-ie\Lambda} \\
&= (\partial_\mu\psi)e^{-ie\Lambda} - ie(\partial_\mu\Lambda)\psi'
\end{aligned}$$

and since

$$\overline{\psi}' = \overline{\psi}e^{ie\Lambda}$$

then

$$\overline{\psi}'\gamma^\mu\partial_\mu\psi' = \overline{\psi}\gamma^\mu\partial_\mu\psi - ie(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi$$

Hence, the Lagrangian density changes as

$$\mathcal{L}' = \mathcal{L} + \frac{i}{2}\left(-ie(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi - ie(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi\right) = \mathcal{L} + e(\partial_\mu\Lambda)\overline{\psi}\gamma^\mu\psi$$

and so is invariant under the combined transformations.

The form of the fourth term can be considered to be fixed by this requirement, so the invariance of the Lagrangian density under the combined transformation determines the coupling of the electron and photon fields.

3. Using the chain rule, then

$$\frac{\partial L}{\partial \phi} = \frac{\partial L}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \phi} + \frac{\partial L}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \phi}$$

The Euler-Lagrange equations then give

$$\begin{aligned}\frac{\partial L}{\partial \phi} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\frac{\partial x}{\partial \phi} + \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}\left(\frac{\partial x}{\partial \phi}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)\frac{\partial y}{\partial \phi} + \frac{\partial L}{\partial \dot{y}}\frac{d}{dt}\left(\frac{\partial y}{\partial \phi}\right) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\frac{\partial x}{\partial \phi} + \frac{\partial L}{\partial \dot{y}}\frac{\partial y}{\partial \phi}\right)\end{aligned}$$

Under the rotation transformations

$$x \to x\cos\phi - y\sin\phi, \qquad y \to x\sin\phi + y\cos\phi$$

then

$$\frac{\partial x}{\partial \phi} = -x\sin\phi - y\cos\phi = -y, \qquad \frac{\partial y}{\partial \phi} = x\cos\phi - y\sin\phi = x$$

Hence also

$$\frac{\partial \dot{x}}{\partial \phi} = -\dot{y}, \qquad \frac{\partial \dot{y}}{\partial \phi} = \dot{x}$$

Note also

$$\frac{\partial L}{\partial x} = -\frac{dV}{dr}\frac{\partial r}{\partial x}$$

where

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left[(x^2+y^2)^{1/2}\right] = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{r}$$

Therefore, Eq. 1 on the problem sheet gives

$$\begin{aligned}\frac{\partial L}{\partial \phi} &= \frac{\partial L}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \phi} + \frac{\partial L}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \phi} \\ &= \left(-\frac{dV}{dr}\frac{x}{r}\right)(-y) + (m\dot{x})(-\dot{y}) + \left(-\frac{dV}{dr}\frac{y}{r}\right)(x) + (m\dot{y})(\dot{x}) \\ &= -\frac{dV}{dr}\left(\frac{-xy}{r} + \frac{xy}{r}\right) + (-m\dot{x}\dot{y} + m\dot{x}\dot{y}) = 0\end{aligned}$$

and so the Lagrangian is indeed invariant under the rotation. The conserved quantity given in Eq. 2 on the problem sheet is

$$\left(\frac{\partial L}{\partial \dot{x}}\frac{\partial x}{\partial \phi} + \frac{\partial L}{\partial \dot{y}}\frac{\partial y}{\partial \phi}\right) = -m\dot{x}y + m\dot{y}x = xp_y - yp_x$$

which is the (orbital) angular momentum in the x, y plane.

Changing variables, then

$$\dot{x} = \frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial \phi}\dot{\phi} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$$

Similarly

$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

Hence

$$\begin{aligned}L &= \frac{1}{2}m\left[(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)^2 + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)^2\right] - V(r) \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)\end{aligned}$$

and so does not depend on ϕ directly. Hence

$$\frac{\partial L}{\partial \phi} = 0, \qquad \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

The Euler-Lagrange equation for ϕ is therefore simply

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0$$

and so is explicitly a conservation law. From the above equations for $\dot{x}$ and $\dot{y}$

$$\dot{x}\sin\phi = \dot{r}\sin\phi\cos\phi - r\dot{\phi}\sin^2\phi, \qquad \dot{y}\cos\phi = \dot{r}\cos\phi\sin\phi + r\dot{\phi}\cos^2\phi$$

so

$$r\dot{\phi} = \dot{y}\cos\phi - \dot{x}\sin\phi$$

and the conserved quantity is

$$mr^2\dot{\phi} = m\left(r\dot{y}\cos\phi - r\dot{x}\sin\phi\right) = mx\dot{y} - my\dot{x} = xp_y - yp_x$$

as before. In terms of r

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - \frac{dV}{dr}, \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

The Euler-Lagrange equation for r is therefore

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{dV}{dr}$$

Let the conserved quantity be L, so

$$mr^2\dot{\phi} = L \qquad \text{and so} \qquad \dot{\phi} = \frac{L}{mr^2}$$

Hence the equation for r is

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{dV}{dr}$$

and so automatically includes the centrifugal force.

4. **This is quite a "convoluted" question and is very optional!!**

The zero component of the four-velocity is

$$\frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \gamma$$

The other three components are

$$\frac{d\boldsymbol{r}}{d\tau} = \frac{dt}{d\tau}\frac{d\boldsymbol{r}}{dt} = \gamma\boldsymbol{\beta}$$

But since

$$E = \gamma m, \qquad \boldsymbol{p} = \gamma m\boldsymbol{\beta}$$

then

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu}{m}$$

From the Lagrangian

$$\frac{\partial L}{\partial\left(dx^\mu/d\tau\right)} = -m\frac{dx_\mu}{d\tau} - qA_\mu$$

so the conserved quantity

$$\begin{aligned}\frac{\partial L}{\partial\left(dx^\mu/d\tau\right)}\frac{dx^\mu}{d\tau} - L &= -m\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} - qA_\mu\frac{dx^\mu}{d\tau} + \frac{1}{2}m\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} + q\frac{dx_\mu}{d\tau}A^\mu \\ &= -\frac{1}{2}m\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = -\frac{1}{2m}p_\mu p^\mu\end{aligned}$$

and hence $p_\mu p^\mu$ is conserved. Using

$$\frac{\partial L}{\partial x_\nu} = \partial^\nu L = -q\frac{dx_\mu}{d\tau}\partial^\nu A^\mu$$

then the Euler-Lagrange equations give

$$-q\frac{dx_\mu}{d\tau}\partial^\nu A^\mu = \frac{d}{d\tau}\left(-m\frac{dx^\nu}{d\tau} - qA^\nu\right) = -\frac{dp^\nu}{d\tau} - q\frac{dx_\mu}{d\tau}\partial^\mu A^\nu$$

and so

$$\frac{dp^\nu}{d\tau} = q\frac{dx_\mu}{d\tau}\partial^\nu A^\mu - q\frac{dx_\mu}{d\tau}\partial^\mu A^\nu = q\frac{dx_\mu}{d\tau}F^{\nu\mu}$$

The zero component of this is

$$\frac{dE}{d\tau} = \gamma\frac{dE}{dt} = q\left(\gamma\beta_x E_x + \gamma\beta_y E_y + \gamma\beta_z E_z\right)$$

or

$$\frac{dE}{dt} = q\boldsymbol{\beta}.\boldsymbol{E}$$

which says the energy of the particle increases as the electric field does work on it. The first component gives

$$\frac{dp_x}{d\tau} = \gamma\frac{dp_x}{dt} = q\left(\gamma E_x + \gamma\beta_y B_z - \gamma\beta_z B_y\right)$$

and similarly for the other two components, so the total can be written as

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{E} + q\boldsymbol{\beta}\times\boldsymbol{B}$$

which is the Lorentz force.

For the Lagrangian including the Λ gauge term

$$\frac{\partial L}{\partial x_\nu} = -q\frac{dx_\mu}{d\tau}\partial^\nu A^\mu - q\frac{dx_\mu}{d\tau}\partial^\nu\partial^\mu\Lambda$$

and

$$\frac{\partial L}{\partial\left(dx^\mu/d\tau\right)} = -m\frac{dx_\mu}{d\tau} - qA_\mu - q\partial_\mu\Lambda$$

so the Euler-Lagrange equation becomes

$$\begin{aligned}-q\frac{dx_\mu}{d\tau}\partial^\nu A^\mu - q\frac{dx_\mu}{d\tau}\partial^\nu\partial^\mu\Lambda &= \frac{d}{d\tau}\left(-m\frac{dx^\nu}{d\tau} - qA^\nu - q\partial^\nu\Lambda\right) \\ &= -\frac{dp^\nu}{d\tau} - q\frac{dx_\mu}{d\tau}\partial^\mu A^\nu - q\frac{dx_\mu}{d\tau}\partial^\mu\partial^\nu\Lambda\end{aligned}$$

The same Λ terms appear on both sides and so cancel, giving no effect. The action here is

$$A = \int L \; d\tau$$

so under a gauge transformation, it changes to

$$A \rightarrow A - \int q \frac{dx_\mu}{d\tau} \partial^\mu \Lambda \; d\tau$$

However, in the same way as for A^μ above, the change of Λ with τ due to the motion of the particle is

$$\frac{d\Lambda}{d\tau} = \frac{\partial \Lambda}{\partial x_\mu} \frac{dx_\mu}{d\tau} = \frac{dx_\mu}{d\tau} \partial^\mu \Lambda$$

so the change in the action is

$$\Delta A = \int q \frac{d\Lambda}{d\tau} \; d\tau = q \, [\Lambda]$$

evaluated at the limits. Hence, for a given function Λ, this is a constant and so has no effect on the position of the minimum of the action, and hence no effect on the particle motion.