Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 3

Questions 1, 2 and if possible 3 should be handed in by 4pm on 8/02/04 for rapid feedback

1. Complex fields have an "internal" symmetry under phase changes, as illustrated in this problem using the example of the Klein-Gordon equation.

   The Lagrangian density for a complex scalar field, ϕ, can be written

   $$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

   Use the Euler-Lagrange equation for ϕ^* to show ϕ satisfies the Klein-Gordon equation.

   Consider the transformation

   $$\phi \to \phi e^{-i\alpha}$$

   for constant α, i.e. $\alpha \neq \alpha(x^\mu)$. Show using

   $$\frac{\partial \mathcal{L}}{\partial \alpha} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\partial \phi^*}{\partial \alpha} \right]$$

   that there is a conserved Nöther current

   $$J^\mu = i \left[\phi^* (\partial^\mu \phi) - (\partial^\mu \phi^*) \phi \right]$$

   Directly prove this current is conserved by differentiating it.

2. Exam question: 2004 question 4.

   The QED Lagrangian density for electrons and photons can be written as

   $$\mathcal{L} = \left[\frac{i}{2} \left(\overline{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \overline{\psi}) \gamma^\mu \psi \right) \right] - \left[m \overline{\psi} \psi \right] - \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] - \left[e A_\mu \overline{\psi} \gamma^\mu \psi \right],$$

   where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $\overline{\psi} = \psi^\dagger \gamma^0$.

   (i) What is the significance of each of the four terms within the square brackets? [4 marks]

   (ii) The Euler-Lagrange equation for any variable q, where $\mathcal{L} = \mathcal{L}(q, \partial_\mu q)$, is

   $$\frac{\partial \mathcal{L}}{\partial q} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \right).$$

   Applying this equation for the case of $q \equiv \overline{\psi}$, obtain the Dirac equation for an electron in an electromagnetic field. Hence, by taking the Hermitian conjugate of this equation, or otherwise, prove the relation

   $$\partial_\mu (e \overline{\psi} \gamma^\mu \psi) = 0.$$

   Explain the significance of this result. You can assume

   $$\gamma^0 \gamma^0 = 1, \qquad \gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0.$$

   [7 marks]

(iii) Show that the gauge transformation

$$A^{\mu\prime} = A^\mu + \partial^\mu \Lambda$$

for any function $\Lambda = \Lambda(x^\mu)$ leaves the field tensor $F^{\mu\nu}$ unchanged and hence show the Lagrangian density is not invariant under this transformation. Obtain an expression for the extra term in the transformed Lagrangian density. [3 marks]

(iv) Show that the Lagrangian density is also not invariant under the local phase transformation

$$\psi' = \psi e^{-ie\Lambda(x^\mu)}.$$

What is the effect of a combined local gauge and phase transformation on the Lagrangian density? Comment on the relevance of this to the form of the final term in the Lagrangian density. [6 marks]

[TOTAL 20 marks]

3. This problem illustrates the connection between rotational symmetry and angular momentum and also shows the use of non-inertial coordinates.

Consider the Lagrangian for a classical non-relativistic particle moving in two dimensions in a central potential

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - V(r)$$

where $\boldsymbol{r} = (x, y)$. For a transformation of these two variables by a parameter ϕ, then

$$\frac{\partial L}{\partial \phi} = \frac{\partial L}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \phi} + \frac{\partial L}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \phi} \tag{1}$$

Show this can be reduced to

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\frac{\partial x}{\partial \phi} + \frac{\partial L}{\partial \dot{y}}\frac{\partial y}{\partial \phi}\right) \tag{2}$$

This Lagrangian is clearly invariant under a rotation in the x, y plane

$$x \to x\cos\phi - y\sin\phi, \qquad y \to x\sin\phi + y\cos\phi$$

Use Eq. 1 to explicitly verify this statement and Eq. 2 to deduce the associated conserved quantity.

Explicitly change variables from x and y to the non-inertial coordinates r and ϕ

$$x = r\cos\phi, \qquad y = r\sin\phi$$

and show the Lagrangian only contains terms in r, $\dot{r}$ and $\dot{\phi}$ but not ϕ. Show the Euler-Lagrange equation for ϕ now explicitly gives a conserved quantity and verify it is the same as the one calculated above.

Show the equation for r can be written in terms of r and its derivatives only and that it automatically acquires a centrifugal term, meaning

$$m\ddot{r} \neq -\frac{dV}{dr}$$

4. **This is quite a "convoluted" question and is optional !!** This problem gives practise with four-vector notation and shows the covariant form of the electromagnetic Lorentz force.

   A relativistic (but not quantum) particle has a path in an inertial frame of $\boldsymbol{r}(t)$, where $d\boldsymbol{r}/dt = \boldsymbol{\beta}$. The proper (instantaneous rest frame) time of the particle, τ, is related to t by $dt = \gamma d\tau$. Hence, show

$$\frac{dx^\mu}{d\tau} = (\gamma, \gamma\boldsymbol{\beta}) = \frac{p^\mu}{m}$$

   which is the four-velocity.

   Consider the following Lagrangian for a particle of charge q moving in an electromagnetic potential A^μ

$$L = -\frac{1}{2}m\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} - q\frac{dx_\mu}{d\tau}A^\mu$$

   Using the equivalent of the energy (Hamiltonian) conservation equation

$$\frac{d}{d\tau}\left[\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\frac{dx^\mu}{d\tau} - L\right] = 0$$

   show that $p^\mu p_\mu$ is conserved.

   Noting that the field A^μ changes with τ because of the motion of the particle

$$\frac{dA^\mu}{d\tau} = \frac{\partial A^\mu}{\partial x_\nu}\frac{dx_\nu}{d\tau} = \frac{dx_\nu}{d\tau}\partial^\nu A^\mu$$

   then use the Euler-Lagrange equations to show

$$\frac{dp^\nu}{d\tau} = q\frac{dx_\mu}{d\tau}F^{\nu\mu}$$

   Write these equations out in three-vector notation.

   This Lagrangian is not invariant under a gauge transformation of A^μ

$$A^\mu \to A^\mu + \partial^\mu\Lambda$$

   However, show that adding this extra term to the Lagrangian, i.e. using

$$L = -\frac{1}{2}m\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} - q\frac{dx_\mu}{d\tau}A^\mu - q\frac{dx_\mu}{d\tau}\partial^\mu\Lambda$$

   has no effect on the above results. Consider the effect of this extra term on the action $A = \int L\, d\tau$ and hence explain why the motion is gauge invariant.