Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 2 Answers

1. Combining the four matrices together gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix}$$

so solving for the α_i gives

$$\alpha_0 = \frac{a+d}{2}, \qquad \alpha_1 = \frac{c+b}{2}, \qquad \alpha_2 = \frac{c-b}{2i}, \qquad \alpha_3 = \frac{a-d}{2}$$

which can always be solved. Hence, any choice for γ^0 can be broken down into these matrices so we try

$$\gamma^0 = \alpha_0 I + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

Taking for example the anticommutator with $\gamma^1 = i\sigma_1$, then

$$\begin{aligned} \gamma^0\gamma^1 + \gamma^1\gamma^0 = 0 &= (\alpha_0 I + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)\, i\sigma_1 + i\sigma_1 (\alpha_0 I + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3) \\ &= 2i\alpha_0\sigma_1 + 2i\alpha_1\sigma_1\sigma_1 + i\alpha_2(\sigma_2\sigma_1 + \sigma_1\sigma_2) + i\alpha_3(\sigma_3\sigma_1 + \sigma_1\sigma_3) \end{aligned}$$

But using the properties of the Pauli matrices

$$\sigma_1\sigma_1 = 1, \qquad \sigma_2\sigma_1 + \sigma_1\sigma_2 = 2\delta_{21} = 0, \qquad \sigma_3\sigma_1 + \sigma_1\sigma_3 = 2\delta_{31} = 0$$

then

$$\gamma^0\gamma^1 + \gamma^1\gamma^0 = 0 = 2i\alpha_0\sigma_1 + 2i\alpha_1$$

for which only a γ^0 with $\alpha_0 = \alpha_1 = 0$ will give the right result. Similarly, by anticommuting with γ^2 and γ^3, then γ^0 also requires $\alpha_2 = 0$ and $\alpha_3 = 0$, respectively, which means $\gamma^0 = 0$. This is not an acceptable solution as $\gamma^0\gamma^0 = 1$. Hence, there is no non-zero γ^0 possible in 2×2 for this choice of the γ^i.

For the 3×3 case, we want to find the trace of the matrices, for example γ^0. We know from the fundamental relation that

$$\gamma^0\gamma^1 + \gamma^1\gamma^0 = 0$$

so

$$\gamma^1\gamma^0\gamma^1 + \gamma^1\gamma^1\gamma^0 = 0$$

Also from the fundamental relation, we have

$$\gamma^1\gamma^1 = -1$$

so

$$\gamma^0 = \gamma^1\gamma^0\gamma^1$$

Taking the trace of both sides

$$Tr(\gamma^0) = Tr(\gamma^1\gamma^0\gamma^1) = Tr(\gamma^1\gamma^1\gamma^0) = -Tr(\gamma^0)$$

and so $Tr(\gamma^0)$ must be zero. The same holds for any of the γ^μ.

The eigenvalues, λ, and eigenvectors, v, of any one of the matrices by definition satisfy

$$\gamma^\mu v = \lambda v$$

Therefore, specifically *not* doing implied summation here

$$\gamma^\mu \gamma^\mu v = \lambda \gamma^\mu v = \lambda^2 v$$

But from the fundamental relation

$$\gamma^\mu \gamma^\mu = g^{\mu\mu}$$

so

$$\lambda = \pm\sqrt{g^{\mu\mu}}$$

or, explicitly

$$\begin{aligned} \lambda &= \pm 1 \quad \text{for} \quad \mu &= 0 \\ \lambda &= \pm i \quad \text{for} \quad \mu &= 1, 2, 3 \end{aligned}$$

Consider γ^0, which has eigenvalues of ± 1. If there are n_+ eigenvalues of $+1$ and n_- of -1, then using the general property of the trace of any matrix

$$Tr(\gamma^0) = \sum_i \lambda_i$$

and the zero value of the trace proved above, then

$$\begin{aligned} n_+ - n_- &= 0 \\ n_+ &= n_- \end{aligned}$$

However, the total number of eigenvalues is equal to the dimension of the matrix, so the dimension of γ^0 is $n_+ + n_- = 2n_+$ and so must be even. A similar argument holds for the other gamma matrices.

2. For the new matrices given by

$$\gamma'^\mu = U \gamma^\mu U^{-1}$$

then

$$\begin{aligned} \gamma'^\mu \gamma'^\nu + \gamma'^\nu \gamma'^\mu &= U\gamma^\mu U^{-1} U \gamma^\nu U^{-1} + U \gamma^\nu U^{-1} U \gamma^\mu U^{-1} \\ &= U \gamma^\mu \gamma^\nu U^{-1} + U \gamma^\nu \gamma^\mu U^{-1} \\ &= U(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) U^{-1} \\ &= 2g^{\mu\nu} U U^{-1} = 2g^{\mu\nu} \end{aligned}$$

also satifies the relation.

The new solution satisfies

$$i\gamma'^\mu \partial_\mu \psi' = m\psi'$$

so

$$iU\gamma^\mu U^{-1} \partial_\mu \psi' = m\psi'$$

or

$$i\gamma^\mu \partial_\mu \left(U^{-1}\psi'\right) = m\left(U^{-1}\psi'\right)$$

so that we can identify $U^{-1}\psi' = \psi$ and so $\psi' = U\psi$.

Taking the Hermitian conjugate of this relation

$$\psi'^\dagger = (U\psi)^\dagger = \psi^\dagger U^\dagger$$

To preserve the normalisation then

$$\psi^\dagger\psi = \psi'^\dagger\psi' = \psi^\dagger U^\dagger U\psi$$

which is clearly true if $U^\dagger = U^{-1}$.

The bilinear combinations contain $\overline{\psi}$, which transforms as

$$\overline{\psi}' = \psi'^\dagger\gamma^{0\prime} = \psi^\dagger U^\dagger U\gamma^0 U^{-1} = \psi^\dagger\gamma^0 U^{-1} = \overline{\psi}U^{-1}$$

so any bilinear combination transforms as

$$\overline{\psi}'\gamma'^\alpha\ldots\gamma'^\zeta\psi' = \overline{\psi}U^{-1}U\gamma^\alpha U^{-1}\ldots U\gamma^\zeta U^{-1}U\psi = \overline{\psi}\gamma^\alpha\ldots\gamma^\zeta\psi$$

and so is not changed by the change in the γ^μ matrices.

3. Consider

$$\left[\hat{H}, \hat{L}_x\right] = \left[-i\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla} + m\gamma^0, -i\left(y\partial_3 - z\partial_2\right)\right]$$

The complication arises because of the $\boldsymbol{\nabla}$ term operating on terms like $y\partial_3$. It is easiest to handle this as separate commutators, so writing $y = x^2$ and $z = x^3$, then

$$\left[\hat{H}, \hat{L}_x\right] = -\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla}, x^2\partial_3\right] + \left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla}, x^3\partial_2\right] - i\left[m\gamma^0, x^2\partial_3\right] + i\left[m\gamma^0, x^3\partial_2\right]$$

Consider the first of these

$$\begin{aligned}
\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla}, x^2\partial_3\right] &= \gamma^0\left(\gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right)\left(x^2\partial_3\right) - \left(x^2\partial_3\right)\gamma^0\left(\gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right) \\
&= x^2\gamma^0\gamma^1\partial_1\partial_3 + \gamma^0\gamma^2\partial_3 + x^2\gamma^0\gamma^2\partial_2\partial_3 + x^2\gamma^0\gamma^3\partial_3\partial_3 \\
&\quad - \left(x^2\partial_3\right)\gamma^0\left(\gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right) \\
&= \left(x^2\partial_3\right)\gamma^0\left(\gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right) + \gamma^0\gamma^2\partial_3 \\
&\quad - \left(x^2\partial_3\right)\gamma^0\left(\gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right) \\
&= \gamma^0\gamma^2\partial_3
\end{aligned}$$

Similarly

$$\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla}, x^3\partial_2\right] = \gamma^0\gamma^3\partial_2$$

The other two commutators are easier

$$-i\left[m\gamma^0, x^2\partial_3\right] = -im\gamma^0 x^2\partial_3 + imx^2\partial_3\gamma^0$$

but as nothing to the right of the derivative depends on the spatial coordinates, then this gives zero. Hence for both

$$-i\left[m\gamma^0, x^2\partial_3\right] = i\left[m\gamma^0, x^3\partial_2\right] = 0$$

This gives

$$\left[\hat{H}, \hat{L}_x\right] = -\gamma^0\gamma^2\partial_3 + \gamma^0\gamma^3\partial_2 = -\gamma^0\left(\boldsymbol{\gamma}\times\boldsymbol{\nabla}\right)_x$$

so generally, for all three components

$$\left[\hat{H}, \hat{\boldsymbol{L}}\right] = -\gamma^0\boldsymbol{\gamma}\times\boldsymbol{\nabla}$$

For the spin operator, then the commutator of the x component with the Hamiltonian is

$$\left[\hat{H},\hat{S}_x\right]=\left[-i\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla}+m\gamma^0,\frac{1}{2}\gamma^5\gamma^0\gamma^1\right]$$

which again can be considered as separate commutators

$$\left[\hat{H},\hat{S}_x\right]=-\frac{i}{2}\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla},\gamma^5\gamma^0\gamma^1\right]+\frac{m}{2}\left[\gamma^0,\gamma^5\gamma^0\gamma^1\right]$$

In this case, there are no spatial components, so $\boldsymbol{\nabla}$ does not operate on anything. The complications arise because the γ matrices do not commute. Consider the first term and pull through the γ^5 matrix, using the fundamental relation

$$\begin{aligned}\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla},\gamma^5\gamma^0\gamma^1\right] &= \gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\gamma^5\gamma^0\gamma^1-\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\\ &= -\gamma^0\gamma^5\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\gamma^0\gamma^1-\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\\ &= \gamma^5\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\gamma^0\gamma^1-\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\end{aligned}$$

Next, pull through the γ^0 and γ^1 matrices

$$\begin{aligned}\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla},\gamma^5\gamma^0\gamma^1\right] &= -\gamma^5\gamma^0\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\gamma^1-\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\\ &= -\gamma^5\gamma^0\gamma^0\gamma^1\left(\gamma^1\partial_1-\gamma^2\partial_2-\gamma^3\partial_3\right)-\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\\ &= \gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^1\partial_1-\gamma^2\partial_2-\gamma^3\partial_3\right)-\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\\ &= -2\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^2\partial_2+\gamma^3\partial_3\right)\end{aligned}$$

This can be reduced to a simpler form. Remembering that $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\gamma^5\gamma^5=1$, then

$$\begin{aligned}\left[\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla},\gamma^5\gamma^0\gamma^1\right] &= -2\gamma^5\gamma^0\gamma^1\gamma^0\left(\gamma^2\partial_2+\gamma^3\partial_3\right)\\ &= -2\gamma^0\gamma^5\gamma^0\gamma^1\gamma^2\partial_2-2\gamma^0\gamma^5\gamma^0\gamma^1\gamma^3\partial_3\\ &= 2\gamma^0\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\partial_2+2\gamma^0\gamma^5\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3\partial_3\\ &= 2\gamma^0\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\partial_2-2\gamma^0\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2\partial_3\\ &= 2\gamma^0\gamma^5(-i\gamma^5)\gamma^3\partial_2-2\gamma^0\gamma^5(-i\gamma^5)\gamma^2\partial_3\\ &= -2i\gamma^0\gamma^3\partial_2+2i\gamma^0\gamma^2\partial_3\\ &= 2i\gamma^0\left(\gamma^2\partial_3-\gamma^3\partial_2\right)\end{aligned}$$

The second commutator is again easier

$$\begin{aligned}\left[\gamma^0,\gamma^5\gamma^0\gamma^1\right] &= \gamma^0\gamma^5\gamma^0\gamma^1-\gamma^5\gamma^0\gamma^1\gamma^0\\ &= -\gamma^5\gamma^0\gamma^0\gamma^1-\gamma^5\gamma^0\gamma^1\gamma^0\\ &= \gamma^5\gamma^0\gamma^1\gamma^0-\gamma^5\gamma^0\gamma^1\gamma^0\\ &= 0\end{aligned}$$

so overall the result is

$$\begin{aligned}\left[\hat{H},\hat{S}_x\right] &= -\frac{i}{2}\left[2i\gamma^0\left(\gamma^2\partial_3-\gamma^3\partial_2\right)\right]\\ &= \gamma^0\left(\gamma^2\partial_3-\gamma^3\partial_2\right)\\ &= \gamma^0\left(\boldsymbol{\gamma}\times\boldsymbol{\nabla}\right)_x\end{aligned}$$

and so, for all three components

$$\left[\hat{H}, \hat{\boldsymbol{S}}\right] = \gamma^0 \boldsymbol{\gamma} \times \boldsymbol{\nabla}$$

It is therefore obvious the sum of the two commutators for $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{S}}$ will give zero, so the total angular momentum is conserved.

4. (i) A parity operation reflects each of the three spatial coordinate axes through the origin, so that $\boldsymbol{r} \rightarrow -\boldsymbol{r}$.

Under a parity operation, a polar vector is any vector which acts in the same way as $\boldsymbol{r}$ above, i.e. each component is reflected. An axial vector has no change of its coordinates under a parity operation. Examples of polar vectors include position $\boldsymbol{r}$, momentum $\boldsymbol{p}$ and the electric field $\boldsymbol{E}$. Examples of axial vectors include angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ (since both $\boldsymbol{r}$ and $\boldsymbol{p}$ change sign) and the magnetic field $\boldsymbol{B}$.

If a system is covariant under a parity operation, then the equations governing it do not change form under such an operation, i.e. they have a symmetry under the parity operation. There is a general connection between such a symmetry and a conservation law. In this case, the symmetry leads to the conservation of a quantity, also called parity. Because the operation is discrete, the allowed values of the parity quantity are also discrete. For a system, then if

$$\hat{P}\psi = P\psi$$

then applying the operation a second time gives

$$\hat{P}\hat{P}\psi = P\hat{P}\psi = P^2\psi$$

However, by definition, applying the operation twice returns the system to the original state, so $P^2 = 1$ and therefore $P = \pm 1$.

(ii) Under a parity operation, the $\boldsymbol{\nabla}$ operator changes sign, so the parity-inverted Dirac equation is

$$i\gamma^0 \partial_0 \psi' - i\boldsymbol{\gamma}.\boldsymbol{\nabla}\psi' - m\psi' = 0,$$

Multiplying from the left by γ^0, then this becomes

$$i\gamma^0\gamma^0 \partial_0 \psi' - i\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla}\psi' - m\gamma^0\psi' = 0$$

Using the properties of the γ matrices

$$\gamma^0\boldsymbol{\gamma} + \boldsymbol{\gamma}\gamma^0 = 0$$

this becomes

$$i\gamma^0 \partial_0 (\gamma^0\psi') + i\boldsymbol{\gamma}.\boldsymbol{\nabla}(\gamma^0\psi') - m(\gamma^0\psi') = 0$$

which is the original Dirac equation with a solution $\gamma^0\psi'$. This must be ψ, so

$$\gamma^0\psi' = \psi$$

and using

$$\gamma^0\gamma^0 = 1$$

then

$$\gamma^0\psi = \gamma^0\gamma^0\psi' = \psi' = \hat{P}\psi$$

(iii) The parity-transformed Hermitian conjugate is

$$\psi'^{\dagger} = (\gamma^0 \psi)^{\dagger} = \psi^{\dagger} \gamma^{0\dagger} = \psi^{\dagger} \gamma^0$$

where $\gamma^{0\dagger} = \gamma^0$ is needed due to the requirement that the Hamiltonian is Hermitian. The parity-transformed adjoint is then

$$\overline{\psi'} = \psi'^{\dagger} \gamma^0 = \psi^{\dagger} \gamma^0 \gamma^0 = \overline{\psi} \gamma^0$$

Hence, under a parity operation, the four-vector J_X^{μ} becomes

$$J_X'^{\mu} = \overline{\psi'} \gamma^{\mu} \phi' = \overline{\psi} \gamma^0 \gamma^{\mu} \gamma^0 \phi$$

Hence, the time and spatial components change as

$$J_X'^{0} = \overline{\psi} \gamma^0 \gamma^0 \gamma^0 \phi = \overline{\psi} \gamma^0 \phi = J_X^0$$

and

$$J_X'^{i} = \overline{\psi} \gamma^0 \gamma^i \gamma^0 \phi = -\overline{\psi} \gamma^i \gamma^0 \gamma^0 \phi = -\overline{\psi} \gamma^i \phi = -J_X^i$$

The time component is unchanged and the spatial components change sign, so this is a polar vector. Similarly, for J_Y^{μ}

$$J_Y'^{\mu} = \overline{\psi'} \gamma^{\mu} \gamma^5 \phi' = \overline{\psi} \gamma^0 \gamma^{\mu} \gamma^5 \gamma^0 \phi = -\overline{\psi} \gamma^0 \gamma^{\mu} \gamma^0 \gamma^5 \phi$$

Hence, the time and spatial components change as

$$J_Y'^{0} = -\overline{\psi} \gamma^0 \gamma^0 \gamma^0 \gamma^5 \phi = -\overline{\psi} \gamma^0 \gamma^5 \phi = -J_Y^0$$

and

$$J_Y'^{i} = -\overline{\psi} \gamma^0 \gamma^i \gamma^0 \gamma^5 \phi = \overline{\psi} \gamma^i \gamma^0 \gamma^0 \gamma^5 \phi = \overline{\psi} \gamma^i \gamma^5 \phi = J_Y^i$$

Here, the time component changes sign and the spatial ones do not, so this is an axial vector.

(iv) Only J_X^{μ} takes part in electromagnetic and strong interactions. Therefore, there is only one type of vector involved and parity is conserved. In weak interactions, both J_X^{μ} and J_Y^{μ} participate and this lack of a definite vector type under parity operations gives rise to the non-conservation of parity in weak interactions.

The classic example of parity non-conservation is the original experiment by Wu. The spins of ^{60}Co atoms were aligned in a magnetic field and the subsequent beta decays observed. The emitted electrons were found to have an angular distribution of $(1 - \beta \cos\theta)$ with respect to the magnetic field. Under a parity transformation, the magnetic field would be unchanged (as it is an axial vector) while the electron momentum would be reversed (as it is a polar vector). The resulting distribution would then be $(1 + \beta \cos\theta)$, which is clearly not covariant.