Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 2

Questions 1, 2 and 3 which support directly the lecture material on the Dirac equation should be handed in by 2pm on 25/01/05 for rapid feedback.

1. This question demonstrates that neither 2×2 nor 3×3 matrices can be used to represent the γ^μ matrices.

   Firstly, show that any 2×2 matrix can be uniquely expressed as a sum of the unit matrix, I, and the Pauli matrices

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha_0 I + \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3$$

   by explicitly solving for the α coefficients and showing a solution is always possible. Hence, taking

   $$\gamma^1 = i\sigma_1 \qquad \gamma^2 = i\sigma_2 \qquad \gamma^3 = i\sigma_3$$

   then show that any choice for γ^0 cannot have the required anticommutator with all these γ^i matrices.

   To eliminate the 3×3 case; using just the fundamental requirement on the γ^μ matrices

   $$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

   and not any explicit representation, show they all must have a trace of zero. Hint: use the cyclicity of matrices in a trace

   $$Tr(ABC) = Tr(CAB) = Tr(BCA)$$

   Also show that the eigenvalues are ± 1 (for $\mu = 0$) and $\pm i$ (for $\mu \neq 0$) and hence show the matrices must have even dimensions, i.e. they cannot be 3×3.

2. The fundamental requirement on the γ^μ matrices is that they satisfy the anticommutator relation, but this does not uniquely define the matrices. This question demonstrates that observables do not depend on the choice of representation.

   Given a set of four γ^μ matrices which obey the fundamental relation, then show a new set of matrices, γ'^μ, given by

   $$\gamma'^\mu = U \gamma^\mu U^{-1}$$

   where U is any invertible matrix, will also satisfy the relation.

   Assume ψ is a solution of the Dirac equation with the original set of matrices. The new set would give a different solution, ψ', where

   $$i\gamma'^\mu \partial_\mu \psi' = m\psi'$$

   Hence show that $\psi' = U\psi$.

   The normalisation of the spinor is defined through $\psi^\dagger \psi$. Show that to preserve the normalisation with the new solution, the matrix U must be unitary, i.e. $U^\dagger = U^{-1}$.

   The various bilinear combinations

   $$\overline{\psi}\psi, \quad \overline{\psi}\gamma^\mu\psi, \quad i\overline{\psi}[\gamma^\mu, \gamma^\nu]\psi/2, \quad \overline{\psi}\gamma^5\gamma^\mu\psi, \quad \overline{\psi}\gamma^5\psi$$

   give physical observables; show that none of these are changed by the choice of the γ^μ matrices.

3. Some details of the proof that the Dirac equation does indeed represent spin 1/2 particles were not given in the lectures.

   First show orbital angular momentum is not conserved. Using the Hamiltonian

$$\hat{H} = -i\gamma^0\boldsymbol{\gamma}.\boldsymbol{\nabla} + m\gamma^0$$

   and the orbital angular momentum operator

$$\hat{\boldsymbol{L}} = -i\boldsymbol{r} \times \boldsymbol{\nabla}$$

   then show that the commutator of these operators is

$$\left[\hat{H}, \hat{\boldsymbol{L}}\right] = -\gamma^0\boldsymbol{\gamma} \times \boldsymbol{\nabla}$$

   Since the proof for the three components of $\hat{\boldsymbol{L}}$ is very similar, it is sufficient to show this for any one component, e.g.

$$\left[\hat{H}, \hat{L}_x\right] = -\gamma^0\left(\gamma^2\frac{\partial}{\partial z} - \gamma^3\frac{\partial}{\partial y}\right) = -\gamma^0\left(\gamma^2\partial_3 - \gamma^3\partial_2\right)$$

   Use just the fundamental relation for the γ^μ matrices, i.e., do not assume a specific representation.

   The total angular momentum must be conserved, so we need a spin operator such that

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$$

   Taking the spin operator as

$$\hat{\boldsymbol{S}} = \frac{1}{2}\gamma^5\gamma^0\boldsymbol{\gamma}$$

   then show that

$$\left[\hat{H}, \hat{\boldsymbol{S}}\right] = \gamma^0\boldsymbol{\gamma} \times \boldsymbol{\nabla}$$

   Again, showing this for a particular component is sufficient.

4. Exam question: 2001 question 4. You should already know what the parity operation is, although it will be covered in more detail later in the course. The answer to section (iv) has not yet been covered in the course.

   (i) Explain what is meant by the parity operation. Explain the terms *polar vector* and *axial vector* and give an example of each from classical physics. Briefly discuss the connection between the covariance of a system under the parity operation and the conservation of the parity quantum number in quantum mechanics. Show that the eigenvalues of parity are ± 1.

   (ii) The Dirac equation can be written as

$$i\gamma^\mu\partial_\mu\psi - m\psi = i\gamma^0\partial_0\psi + i\boldsymbol{\gamma}.\boldsymbol{\nabla}\psi - m\psi = 0,$$

   where the γ^μ matrices satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}.$$

   Write down the Dirac equation after a parity inversion, assuming that, if ψ is a solution of the original equation, then $\psi' = \hat{P}\psi$ is a solution of the parity-inverted equation. Hence, show that the parity operator can be written as $\hat{P}\psi = \gamma^0\psi$.

(iii) Within the Standard Model, all interactions take place through the four-vector currents

$$J_X^\mu = \overline{\psi}\gamma^\mu\phi, \qquad J_Y^\mu = \overline{\psi}\gamma^\mu\gamma^5\phi,$$

where $\overline{\psi} = \psi^\dagger\gamma^0 = (\gamma^0\psi)^\dagger$ and the matrix γ^5 satisfies

$$\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0.$$

Determine how the time and spatial components of these currents change under a parity operation and hence state whether they are polar or axial vectors.

(iv) Which of the above currents participate in the electromagnetic, strong and weak interactions? Explain why this means the weak interactions do not conserve parity and give an example of a measurement which demonstrates parity is not conserved.