Advanced Particle Physics 04/05
Dr Gavin Davies - Problem Sheet 1 Answers

1. The Lorentz transformations along the z axis for t and z are

$$t' = \gamma(t - \beta z), \qquad z' = \gamma(z - \beta t)$$

The same equations therefore hold for E and p_z. In the rest frame (R), the values are $E_R = m$ and $p_{Rx} = p_{Ry} = p_{Rz} = 0$. Boosting the observer by $-\beta$ so that the particle appears to be moving by $+\beta$ along the z axis, then the boosted values are given by

$$E = \gamma(E_R + \beta p_{Rz}) = \gamma m$$

and

$$p_z = \gamma(p_{Rz} + \beta E_R) = \gamma\beta m$$

Clearly, the boosted values of $p_x = p_y = 0$ as these are not affected by a Lorentz transformation. Hence, the magnitude of the momentum is given by

$$p = \gamma m \beta$$

Noting that this gives $p = E\beta$, then inverting these equations simply gives

$$\gamma = \frac{E}{m}, \qquad \beta = \frac{p}{E}$$

The initial value of $E_R^2 - p_R^2$ is simply m^2. The boosted value is

$$E^2 - p^2 = \gamma^2 m^2 - \gamma^2 m^2 \beta^2 = \gamma^2 m^2 (1 - \beta^2)$$

But since

$$\gamma^2 = \frac{1}{1 - \beta^2}$$

then

$$E^2 - p^2 = m^2$$

and so this combination is invariant. Physically, it corresponds to the square of the particle mass.

2. The total energy and momentum are conserved, so in the rest frame of X then

$$m_X = E_c + E_d, \qquad \boldsymbol{p}_d = -\boldsymbol{p}_c$$

The latter gives

$$p_c^2 = p_d^2$$

which can be expanded as

$$E_c^2 - m_c^2 = E_d^2 - m_d^2$$

and can be rearranged, using the difference of squares, to give

$$m_c^2 - m_d^2 = E_c^2 - E_d^2 = (E_c + E_d)(E_c - E_d)$$

so, using the energy conservation equation above

$$E_c - E_d = \frac{m_c^2 - m_d^2}{m_X}$$

Summing gives

$$2E_c = m_X + \frac{m_c^2 - m_d^2}{m_X}$$

so

$$E_c = \frac{m_X^2 + m_c^2 - m_d^2}{2m_X}$$

and similarly for E_d

$$E_d = \frac{m_X^2 + m_d^2 - m_c^2}{2m_X}$$

Check energy conservation by summing these to give

$$E_c + E_d = \frac{m_X^2 + m_c^2 - m_d^2}{2m_X} + \frac{m_X^2 + m_d^2 - m_c^2}{2m_X} = \frac{2m_X^2}{2m_X} = m_X$$

If the final state particles are the same, or antiparticles of each other, then $m_c = m_d$ so

$$E_c = \frac{m_X^2}{2m_X} = \frac{m_X}{2} = E_d$$

and each particle has half the available energy, as expected by symmetry.

The magnitude of the momentum is given by

$$p_c = \sqrt{E_c^2 - m_c^2} = \sqrt{\frac{(m_X^2 + m_c^2 - m_d^2)^2}{4m_X^2} - m_c^2}$$

This becomes

$$p_c = \frac{\sqrt{m_X^4 + m_c^4 + m_d^4 + 2m_X^2 m_c^2 - 2m_X^2 m_d^2 - 2m_c^2 m_d^2 - 4m_X^2 m_c^2}}{2m_X}$$

which is

$$p_c = \frac{\sqrt{m_X^4 + m_c^4 + m_d^4 - 2m_X^2 m_c^2 - 2m_X^2 m_d^2 - 2m_c^2 m_d^2}}{2m_X}$$

This is symmetric for c and d so the value of p_d is the same; this is obviously necessary as the magnitudes of the momenta are by definition the same in the centre-of-mass.

If $m_c = m_d$, then this reduces to

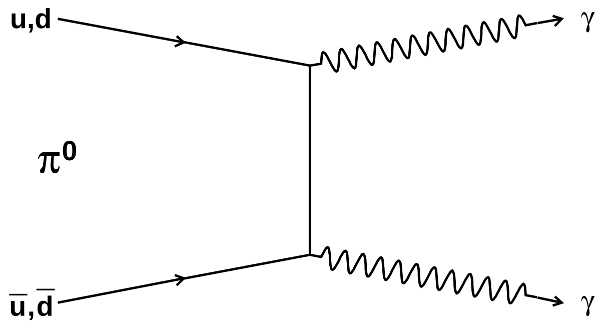
$$p_c = \frac{\sqrt{m_X^4 - 4m_X^2 m_c^2}}{2m_X} = \sqrt{\frac{m_X^2}{4} - m_c^2}$$

as would be expected from the result for the energy above.

Finally, if $m_d = 0$, then

$$p_c = \frac{\sqrt{m_X^4 + m_c^4 - 2m_X^2 m_c^2}}{2m_X} = \frac{\sqrt{(m_X^2 - m_c^2)^2}}{2m_X} = \frac{m_X^2 - m_c^2}{2m_X}$$

3. (i) The Feynman diagram for $\pi^0 \to \gamma\gamma$ is shown below.
In the π^0 rest frame, the photon energies must be equal and have a magnitude $E_\gamma = m_\pi/2 = 67.5$ MeV.



(ii) For a 0.5 GeV π^0, $\gamma = E_\pi/m_\pi = 3.70$ and $\beta = p_\pi/E_\pi = \sqrt{E_\pi^2 - m_\pi^2}/E_\pi = 0.963$ so the energy of a photon emitted in the same direction as the π^0 will be

$$E'_1 = \gamma(E_\gamma + \beta p_\gamma) = \gamma(1+\beta)E_\gamma = 0.490 \text{ GeV}$$

which is therefore the maximum possible energy. The second photon in this situation will have an energy $E'_2 = 0.5 - E'_1 = 0.010$ GeV, also given by

$$E'_2 = \gamma(1-\beta)E_\gamma = 0.010 \text{ GeV}$$

and it will be going in the opposite direction to the first photon.

(iii) For back-to-back photons, then the total momentum magnitude is given by

$$(\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 = (E'_1 - E'_2)^2$$

and since the total energy is $E'_1 + E'_2$ then the invariant mass is

$$m = \sqrt{(E'_1 + E'_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2} = \sqrt{(E'_1 + E'_2)^2 - (E'_1 - E'_2)^2}$$

which simplifies to

$$= \sqrt{E_1'^2 + E_2'^2 + 2E'_1E'_2 - E_1'^2 - E_2'^2 + 2E'_1E'_2} = 2\sqrt{E'_1E'_2}$$

(iv) The partial derivatives of the invariant mass are

$$\frac{\partial m}{\partial E'_1} = \frac{E'_2}{\sqrt{E'_1E'_2}} = \sqrt{\frac{E'_2}{E'_1}}$$

and similarly for E'_2. Hence, the error on the invariant mass is

$$\sigma_m^2 = \left(\frac{\partial m}{\partial E'_1}\right)^2 \sigma_{E'_1}^2 + \left(\frac{\partial m}{\partial E'_2}\right)^2 \sigma_{E'_2}^2$$

and with

$$\sigma_{E'_i} = 0.02E'_i$$

then

$$\sigma_m^2 = \frac{E'_2}{E'_1}(0.02)^2E_1'^2 + \frac{E'_1}{E'_2}(0.02)^2E_2'^2 = 2(0.02)^2E'_1E'_2$$

and so

$$\sigma_m = 0.02\sqrt{2E'_1E'_2} = 0.02\frac{m}{\sqrt{2}} = 1.9 \text{ MeV}$$

In natural units, the π^0 lifetime is 8.4×10^{-17} s $\times 1.519 \times 10^{24}$ GeV^{-1}s^{-1} $= 1.28 \times 10^8$ GeV^{-1} so the width is $\Gamma_\pi = 1/\tau_\pi = 7.9 \times 10^{-9}$ GeV or 7.9×10^{-6} MeV, which is much smaller than the width due to experimental resolution. Hence, the observed width is completely due to the calorimeter performance.

(v) For a 4 GeV π^0, $\gamma = E_\pi/m_\pi = 29.6$ and $\beta = p_\pi/E_\pi = \sqrt{E_\pi^2 - m_\pi^2}/E_\pi = 0.9994$. In the rest frame of the π^0, the average decay time is τ_π. In the laboratory frame, time dilation means the average decay time is lengthened to $\langle t \rangle = \gamma\tau_\pi$. During this time, the π^0 will have gone a distance $\langle t \rangle c\beta$ so the average decay length is

$$\langle l \rangle = \langle t \rangle c\beta = \gamma\beta c\tau_\pi = 7.45 \times 10^{-7} \text{ m}$$

which is a negligible distance compared with the dimensions of the calorimeter.

The longitudinal momentum of the photons in the laboratory frame is given by

$$p_l' = \gamma(p_{l\gamma} + \beta E_\gamma) = \gamma\beta E_\gamma = 1.9989 \text{ GeV}$$

while their transverse momentum is unchanged by the π^0 motion and so is $p_t = p_t' =$ 67.5 MeV. The total opening angle, α, between them is therefore

$$\alpha = 2\tan^{-1}\left(\frac{p_t'}{p_l'}\right) = 3.9^\circ = 67 \text{ mrad}$$

The distance between the photons when they hit the calorimeter face is $r\alpha = 6.7$ cm, which is of the same order as the size of each crystal face. Therefore, the photons will often be in the same crystal, and always at least in neighbouring crystals, so that they cannot easily be distinguished as separate photons. Hence, no invariant mass can be calculated and so high energy π^0 mesons cannot easily be detected using this technique at BaBar.

4. Using the Lorentz transformation for the momentum

$$p_c' = \gamma(p_c + \beta E_c) = \gamma\left[\sqrt{\left(\frac{m_X^2}{4} - m_c^2\right)} + \beta\frac{m_X}{2}\right]$$

This is larger than the momentum of X when $p_c' > \gamma\beta m_X$ which corresponds to

$$\gamma\left[\sqrt{\left(\frac{m_X^2}{4} - m_c^2\right)} + \beta\frac{m_X}{2}\right] > \gamma\beta m_X$$

or

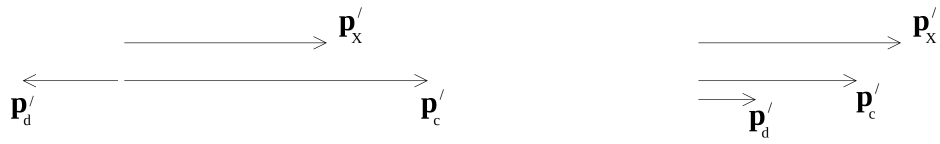
$$\sqrt{\left(\frac{m_X^2}{4} - m_c^2\right)} > \beta\frac{m_X}{2}$$

In terms of the rest frame momentum and energy, this is

$$p_c > \beta E_c \quad \text{so} \quad \frac{p_c}{E_c} = \beta_c > \beta$$

Therefore, the requirement is that the speed of particle c in the rest frame is greater than the speed of the particle X in the moving frame, i.e. the boost velocity.

This is straightforward to understand using momentum conservation; when $p_c' > p_X'$, then particle d must be moving in the opposite direction to balance momentum. This means in



the boosted frame, it must have a velocity along the $-z$ axis which means its speed in the rest frame (which is the same as that of c) must be bigger than that of X, i.e. the boost velocity. Conversely, if $p'_c < p'_X$, then the boost velocity is bigger than the speed of d and so it gets completely flipped around and is going in the same direction as c.

For $m_c = m_d = 0$, then $p_c = E_c = m_X/2$. The Lorentz transformations on E_c and p_{cz} are then

$$E'_c = \gamma(E_c + \beta p_{cz}) = \gamma\left(\frac{m_X}{2} + \beta\frac{m_X}{2}\cos\theta_c\right)$$

Hence

$$dE'_c = \frac{\gamma\beta m_X}{2} d(\cos\theta_c)$$

and so the probability distribution for the energy is

$$P(E'_c)dE'_c = \frac{1}{\gamma\beta m_X} dE'_c$$

This is also a flat distribution (i.e. does not depend on energy) with limits given by $\cos\theta_c = \pm 1$, which are

$$E'_{c\ \mathrm{max/min}} = \gamma\left(\frac{m_X}{2} \pm \beta\frac{m_X}{2}\right) = \frac{\gamma m_X}{2}(1 \pm \beta) = \frac{E'_X}{2}(1 \pm \beta)$$