Lecture 6, February 6th 2007

Solutions of the Dirac Equation

In the previous lecture we used the Dirac equation to derive the continuity and the current equation for spinors and showed that the Dirac equation always leads to states which have probability greater or equal to zero. Hence, it does not suffer from the negative probability problems of the Klein Gordon equation. In this lecture we solve the Dirac equation and show that like the Klein Gordon equation it has both positive and negative energy solutions. Some of the properties of the solutions of the Dirac equation are also presented.

The reader may wonder what are we going to do with the negative energy solutions. As it will be shown later, Dirac solved this problem by proposing his hole-theory, valid only for fermions, according to which the negative energy solutions are re-interpreted as antiparticle solutions. Eventually all the negative energy solutions for both Bosons and Fermions were re-interpreted by Feynman and Stückelberg in a consistent framework which will be the subject of a next lecture.

Solutions of the Dirac Equation

We start from the Dirac equation in the covariant form:

$$[i\gamma^{\mu}\partial_{\mu}-m]\Psi(x) = 0 \qquad (1)$$

The anti-commutation relations form the previous lecture:

$$\{\alpha_i,\alpha_j\} = 2\delta_{ij} \ , \ \{\beta,\alpha_j\} = 0 \ , \quad \beta^2 = 1$$

combined with the definitions of the gamma matrices:

$$\gamma^i = \beta\alpha^i \quad i = 1,2,3 \ ; \ \gamma^0 = \beta \ ; \quad \gamma^{\mu} = (\gamma^0;\vec{\gamma})$$

lead the to covariant anti-commutation relations for the gamma matrices:

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \qquad (2)$$

Assume that the solutions to the Dirac equation are of the form:

$$\Psi(x^{\mu}) = u(\vec{p})\mathrm{e}^{-ip^{\mu}\cdot x_{\mu}} \qquad (3)$$

From (1) and (3) we have that: $[\gamma^{\mu}p_{\mu}-m]\Psi(x) = 0 \qquad (4)$

Equation (4) can be written in a matrix form as:

$$\left[\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{pmatrix} \boldsymbol{p}^{0} - \begin{pmatrix} \boldsymbol{0} & \vec{\sigma} \\ -\vec{\sigma} & \boldsymbol{0} \end{pmatrix} \cdot \vec{\boldsymbol{p}} - \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \boldsymbol{m}\right] \boldsymbol{u}(\vec{\boldsymbol{p}}) = \boldsymbol{0} \qquad (5)$$

where the exponential term is not needed and has been omitted. It is perhaps worth reminding the reader that although the matrix equation above seems to be a 2x2 matrix equation, in reality it refers to 4x4 matrix objects except for $\boldsymbol{u}(\vec{\boldsymbol{p}})$ which is a 4x1 column object.

We can write the spinor $\boldsymbol{u}(\vec{\boldsymbol{p}})$ in terms of arbitrary χ, ϕ 2x1 column matrices as:

$$\boldsymbol{u}(\vec{\boldsymbol{p}}) = \begin{pmatrix} \chi \\ \phi \end{pmatrix} \qquad (6)$$

Substituting (6) into (5) we get: $\begin{pmatrix} \boldsymbol{p}^{0} - \boldsymbol{m} & -\vec{\sigma}\cdot\vec{\boldsymbol{p}} \\ \vec{\sigma}\cdot\vec{\boldsymbol{p}} & -(\boldsymbol{p}^{0} + \boldsymbol{m}) \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \boldsymbol{0} \qquad (7)$

If (7) is to have non-zero (non-trivial) solutions, the determinant of the matrix multiplying $\boldsymbol{u}(\vec{\boldsymbol{p}})$ must be zero (so that the inverse matrix does not exist). Hence:

$$(\boldsymbol{p}^{0} - \boldsymbol{m})(-\boldsymbol{1})(\boldsymbol{p}^{0} + \boldsymbol{m}) + (\vec{\sigma}\cdot\vec{\boldsymbol{p}})^{2} = \boldsymbol{0} \qquad (8)$$

and using the identity: $(\vec{\sigma}\cdot\vec{\boldsymbol{p}})^{2} = \vec{\boldsymbol{p}}^{2}$, equation (8) gives:

$$(\boldsymbol{p}^{0})^{2} = \vec{\boldsymbol{p}}^{2} + \boldsymbol{m}^{2} \Rightarrow \boldsymbol{p}^{0} = \pm\sqrt{\vec{\boldsymbol{p}}^{2} + \boldsymbol{m}^{2}} \Rightarrow \boldsymbol{p}^{0} = \boldsymbol{E} = \pm\sqrt{\vec{\boldsymbol{p}}^{2} + \boldsymbol{m}^{2}} \qquad (9)$$

Hence, $\boldsymbol{p}^{0}$ can be identified with the relativistic energy of the particle. **However, as in the Klein Gordon equation we have both positive and negative energy solutions.** Lets ignore this for the moment and continue solving the Dirac equation.

Equations (8) and (9) give: $$\begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m) \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix} = 0 \qquad (10)$$

First lets try to solve the Dirac equation at the rest frame of the particle where the momentum is zero:

<u>Positive Energy Solutions</u>

If $E=+\sqrt{m^2}>0$ from (10) we have that:

$$\begin{pmatrix} 0 & 0 \\ 0 & -(2m) \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix} = 0$$

This means that for positive energy solutions we have that $\phi = 0$ and $\chi \neq 0$.

In other words in general we could have that: $\chi = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Clearly there are two independent solutions of the form:

$$\Psi^1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \quad \text{and} \quad \Psi^2(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}$$

Negative Energy Solutions

Next we deal with the negative solutions where $E=-\sqrt{m^2}<0$. In this case eq. (10) gives:

$$\begin{pmatrix} -2m & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} = 0$$

which means that $\chi = 0$ and $\phi \neq 0$. As before:

$$\phi = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and we have two more negative energy independent solutions :

$$\Psi^3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{+imt} \quad \text{and} \quad \Psi^4(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{+imt}$$

The student who knows quantum mechanics will have realized by now that **there is a two-fold degeneracy in the energy spectrum**. **For every energy eigenvalue we have two eigenvectors orthogonal to each other. Hence, there must be an operator that commutes with the Hamiltonian which has a common set of eigenvectors with the Hamiltonian**. We will revisit this issue later when we discuss about helicity.

Next lets try to solve the Dirac equation in the general case where the momentum is not zero. Equation (10) gives:

$$(E-m)\chi - (\vec{\sigma}\cdot\vec{p})\phi = 0 \ \Rightarrow \ \chi = \frac{(\vec{\sigma}\cdot\vec{p})}{(E-m)}\phi \quad (11)$$

$$(\vec{\sigma}\cdot\vec{p})\chi - (E+m)\phi = 0 \ \Rightarrow \ \phi = \frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\chi \quad (12)$$

Positive Energy Solutions

Equation (12) is well defined for positive energy solutions (the denominator is always non-zero). Hence the 4x1 column solutions can be written as:

$$\Psi^{(1,2)}(\boldsymbol{x}) = \boldsymbol{N}\begin{pmatrix} \chi^{\pm} \\ \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})}\chi^{\pm} \end{pmatrix}\mathbf{e}^{-ip^{\mu}x_{\mu}} = \boldsymbol{N}\begin{pmatrix} \mathbf{1} \\ \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})} \end{pmatrix}\chi^{\pm}\mathbf{e}^{-ip^{\mu}x_{\mu}} \quad (13)$$

where $\chi^{+} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ and $\chi^{-} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$. The superscript (1) indicates the first solution obtained using χ^{+} and the superscript (2) indicates the second solution obtained using χ^{-}. Note that $\vec{\sigma}\cdot\hat{\boldsymbol{p}}\chi^{\pm} = \pm\chi^{\pm}$ where $\hat{\boldsymbol{p}}$ is the unit vector at the direction of the momentum of the particle. It is convenient but not necessary to choose the particle direction to be along the z-axis as done here. **N** is a normalization constant to be fixed later.

Negative Energy Solutions

Similarly for negative energies only (11) is well defined and we use it to obtain the spinors corresponding to the negative energy

$$\Psi^{(3,4)}(\boldsymbol{x}) = \boldsymbol{N}\begin{pmatrix} \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}-\boldsymbol{m})}\chi^{\pm} \\ \chi^{\pm} \end{pmatrix}\mathbf{e}^{-ip^{\mu}x_{\mu}} = \boldsymbol{N}\begin{pmatrix} \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}-\boldsymbol{m})} \\ \mathbf{1} \end{pmatrix}\chi^{\pm}\mathbf{e}^{-ip^{\mu}x_{\mu}} \quad (14)$$

It is worth noting here that this form of solutions is not unique. For example had we started by assuming a solution of the form:

$$\Psi(\boldsymbol{x}^{\mu}) = \boldsymbol{v}(\vec{\boldsymbol{p}})\mathbf{e}^{+i\,p^{\mu}\cdot x_{\mu}}$$

(note that the sign of the exponent is now positive instead of negative) then we would have obtained as solutions:

Positive Energy Solutions:

$$\Psi^{(1,2)}(x) = N\begin{pmatrix} \frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\chi^{\pm} \\ \chi^{\pm} \end{pmatrix} e^{+ip^{\mu}x_{\mu}} = N\begin{pmatrix} \frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)} \\ 1 \end{pmatrix}\chi^{\pm} e^{+ip^{\mu}x_{\mu}} \qquad (15)$$

Negative Energy Solutions:

$$\Psi^{(3,4)}(x) = N\begin{pmatrix} \chi^{\pm} \\ \frac{(\vec{\sigma}\cdot\vec{p})}{(E-m)}\chi^{\pm} \end{pmatrix} e^{-ip^{\mu}x_{\mu}} = N\begin{pmatrix} 1 \\ \frac{(\vec{\sigma}\cdot\vec{p})}{(E-m)} \end{pmatrix}\chi^{\pm} e^{-ip^{\mu}x_{\mu}} \qquad (16)$$

Note that equations (14) and (15), that is the negative energy solutions of the first set and the positive energy solutions of the second set, are related by the a simple transformation where: $p^{\mu} \rightarrow p^{\mu\prime} = -p^{\mu}$. This is not an accident and as we shall see later we will interpret (13) as a positive energy electron solution and (15) as a positive energy positron solution. **In other words negative energy particle solutions, like (14), going backward in time are equivalent with positive energy anti-particle solutions propagating forward in time.**

The Dirac Current and Normalization of the Dirac Solutions

As we have seen before the 0th component of the current density is the particle probability density and the other three components represent the 3-dimensional particle current density:

$$J^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x) \Rightarrow \rho = \bar{\Psi}\Psi = \Psi^{+}\gamma^{0}\gamma^{0}\Psi = \Psi^{+}\Psi$$

$$\Rightarrow \vec{J} = \Psi^{+}\gamma^{0}\vec{\gamma}\Psi$$

We are going to use this to derive the normalization, **N**, of the Dirac spinors:

$$\rho \;=\; J^0 \;=\; \bar{\Psi}\Psi \;=\; |N|^2 \chi^{s+} \left(\mathbf{1}, \quad \frac{(\vec{\sigma}^+ \cdot \vec{p})}{(E+m)} \right) \begin{pmatrix} \mathbf{1} \\ \dfrac{(\vec{\sigma}\cdot\vec{p})}{(E+m)} \end{pmatrix} \chi^s$$

$$\rho \;=\; J^0 \;=\; |N|^2 \chi^{s+} \left(\mathbf{1} + \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2} \right) \chi^s \;=\; \frac{2E}{E+M} |N|^2 \geqslant \mathbf{0}$$

Hence, we have verified explicitly that that probability is positive. As seen before these solutions must normalize to **2E particles per unit volume** (the probability must transform as the 0th component of a 4-vector as seen in the Klein Gordon case) which means that:

$$N = \sqrt{E+m} \quad \text{and} \quad \rho \;=\; +2E \tag{17}$$

Similarly for negative energy solutions we have that:

$$\rho \;=\; J^0 \;=\; |N|^2 \chi^{s+} \left(\frac{(\vec{\sigma}^+ \cdot \vec{p})}{(E-m)}, \quad \mathbf{1} \right) \begin{pmatrix} \dfrac{(\vec{\sigma}\cdot\vec{p})}{(E-m)} \\ \mathbf{1} \end{pmatrix} \chi^s \;=\; |N|^2 \left(\frac{(\vec{p})^2}{(E-m)^2} + \mathbf{1} \right)$$

$$\rho \;=\; J^0 \;=\; |N|^2 \frac{2E}{(E-M)} \;=\; |N|^2 \frac{2|E|}{(|E|+M)} \geqslant \mathbf{0}$$

which gives:

$$N = \sqrt{|E|+m} \text{ and } \rho \;=\; +2|E| \tag{18}$$

Lets now calculate the 3-d vector current using positive energy spinors:

$$\vec{J} = \bar{\Psi}(x)\vec{\gamma}\Psi(x) \Rightarrow$$

$$\vec{J} = (E+m)(\chi^s)^+ \left(1, \ \frac{(\vec{\sigma}^+\cdot\vec{p})}{(E+m)}\right)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)} \end{pmatrix}\chi^s \Rightarrow$$

$$\vec{J} = (E+m)\left((\chi^s)^+, \ \ (\chi^s)^+\frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\right)\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \chi^s \\ \frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\chi^s \end{pmatrix} \Rightarrow$$

$$\vec{J} = (E+m)\left((\chi^s)^+, \ \ (\chi^s)^+\frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\right)\begin{pmatrix} \vec{\sigma}\frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\chi^s \\ \vec{\sigma}\chi^s \end{pmatrix} \Rightarrow$$

$$\vec{J} = (E+m)\left((\chi^s)^+\vec{\sigma}\frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\chi^s + (\chi^s)^+\frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\right)(\vec{\sigma}\chi^s) \Rightarrow$$

It is now more convenient to write the current in terms of its components (with i=1,2,3)

$$J^i = (E+m)\left((\chi^s)^+\sigma^i\frac{(\sigma^j p^j)}{(E+m)}\chi^s + (\chi^s)^+\frac{(\sigma^j p^j)}{(E+m)}\sigma^i\chi^s\right) \Rightarrow$$

$$J^i = (E+m)\left((\chi^s)^+\sigma^i\frac{(\sigma^j p^j)}{(E+m)}\chi^s + (\chi^s)^+\frac{(\sigma^j p^j)}{(E+m)}\sigma^i\chi^s\right) \Rightarrow$$

$$J^i = (\chi^s)^+\left(\sigma^i(\sigma^j p^j)+(\sigma^j p^j)\sigma^i\right)\chi^s \Rightarrow \quad J^i = p^j(\chi^s)^+\left(\sigma^i\sigma^j+\sigma^j\sigma^i\right)\chi^s \Rightarrow$$

The Pauli matrices have the property: $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$. Hence, the current becomes:

$$J^i = p^j(\chi^s)^+ 2\delta^{ij}\chi^s \Rightarrow \quad J^i = 2p^i \Rightarrow \vec{J} = 2\vec{p} \qquad (19)$$

From (17) and (19) we have that the covariant current is:

$$J^\mu = 2p^\mu = 2(E;\vec{p})$$

By multiplying by (**-e**) one can convert this to the electromagnetic charge and current density for electrons as :

$$\rho_{EM} = -2eE = -2ep^0 \quad ; \quad e > 0$$

$$\vec{J}_{EM} = -2e\vec{p}$$

$$J^\mu_{EM} = -2e\,p^\mu \qquad (20)$$

When we discuss local gauge invariance we will see that this factor of charge comes in a more natural way in to the current equation. For the moment we just include it 'by-hand'. So equation (20) gives us the electromagnetic current for positive energy electrons (this is how we constructed it from the positive energy solutions).

One point to be made here is that if equation (20) give us the electromagnetic current density for electrons then:

$$J^\mu_{EM} = +2e\,p^\mu \qquad (21)$$

must be the one for positive energy positrons. Equation (21) can be written as:

$$J^\mu_{EM} = +2e\,p^\mu = -2e(-p^\mu) = -2e(-E;-\vec{p}) \qquad (22)$$

However, equation (22) looks very much like the current for electrons given by (20) but with the signs of energy and momentum reversed. **So it appears that the negative energy electron solutions may be used this way to describe positive energy positrons. In other words in QFT language: emitting (creating) a positive energy positron is equivalent to absorbing (annihilating) a negative energy electron.** Again, this is consistent with the comments we made when we were discussing equation (15) and we will come back to tall these later when we discuss antiparticles.

Spin and Helicity of the Dirac Solutions

Define the spin operator as $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & \mathbf{0} \\ \mathbf{0} & \vec{\sigma} \end{pmatrix}$ and the helicity operator as

$$\vec{\Sigma}\cdot\hat{\boldsymbol{p}} = \begin{pmatrix} \vec{\sigma}\cdot\hat{\boldsymbol{p}} & \mathbf{0} \\ \mathbf{0} & \vec{\sigma}\cdot\hat{\boldsymbol{p}} \end{pmatrix}$$

where $\hat{\boldsymbol{p}}$ is the unit vector at the direction of the particle momentum.

It is easy and it is left for homework to show that the helicity operator commutes with the Hamiltonian $[\vec{\Sigma}\cdot\hat{\boldsymbol{p}}, \boldsymbol{H}] = \mathbf{0}$. **In other words helicity is a conserved quantity**. However since it is expressed in a 3-d vector product, **helicity is not a Lorentz invariant.** The reason for this is easy to understand: Helicity is the projection of the spin at the direction of motion. Consider an observer that moves faster than a given particle of a definite helicity. The observer overcomes the particle and in his frame he starts seeing it moving away from him. In other words the momentum of the particle has flipped sign as far as he is concerned. However, the spin does not flip sign (why should it any way ?). Hence, the moving observer, when he overcomes the particle, sees that the particle helicity has changed sign. Therefore, although the helicity can be conserved in a given frame (commutes with the Hamiltonian), it is not Lorentz invariant (Its value changes from frame to frame). The careful reader must have noticed that for the above argument to hold, it must be that the particle has some mass hence it does not travel with the speed of light and there is always the possibility of being overcome by something faster. If the particle has no mass and therefore moves with the speed of light, then it is impossible to find an observer that overcomes it and the argument is no longer valid. Hence, massless particles do not flip helicity form frame to frame which gives us an indication that helicity must be somehow Lorentz invariant when it come to massless particles. It turns out that it a a bit more complicated than that. The answer to this will be given in one of the next Lectures when we discuss Chirality and Helicity.

It is important to understand that **the solutions of the Dirac equation are not eigenfunctions of the spin operator but only of the helicity operator.** In other words only the spin at the direction of motion is a good quantum number which is conserved.

This of course comes from the fact that the helicity operator commutes with the Hamiltonian and explains the origin of the two-fold degeneracy of the Dirac solutions discussed before. Each of the solutions appears with both positive and negative helicity corresponding to the same energy.

The reader may be surprised when he finds out that neither the spin nor the angular momentum are independently conserved since it can be shown that:

$$[\vec{\Sigma}, \boldsymbol{H}] \neq \mathbf{0} \text{ and } [\vec{\boldsymbol{L}}, \boldsymbol{H}] \neq \mathbf{0}$$

However, t**he *total angular momentum,***

$$\vec{\boldsymbol{J}} = \vec{\boldsymbol{L}} + \frac{\mathbf{1}}{\mathbf{2}}\vec{\Sigma}$$

is conserved because it can be shown that:

$$[\boldsymbol{H}, \vec{\boldsymbol{L}} + \frac{\mathbf{1}}{\mathbf{2}}\vec{\Sigma}] = \mathbf{0}$$

This result gives us a strong indication that the Dirac equation describes spin half fermions. However, showing this will be left for the homework.