Particle Physics Homework Assignment 5

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Problem 1: As shown in class the Dirac matrices must satisfy the anti-commutator relationships:

$$\{\alpha_i, \alpha_j\} = \mathbf{2}\delta_{ij}, \quad \{\alpha_i, \beta\} = \mathbf{0} \quad \text{with} \quad \beta^2 = \mathbf{1}$$

I. Show that the α_i, β are Hermitian, traceless matrices with eigenvalues $\pm\mathbf{1}$ and even dimensionality.

II. Show that, as long as the mass term is not zero and the matrix β is needed, there is no 2x2 set of matrices that satisfy all the above relationships. Hence, the Dirac matrices must be of dimension 4 or higher. First show that the set of matrices $(\mathbf{1};\vec{\sigma})$ can be used to express any $\mathbf{2}\times\mathbf{2}$ matrix. That is the coefficients $\boldsymbol{c_0}, \boldsymbol{c_i}$ always exist such that any $\mathbf{2}\times\mathbf{2}$ matrix can be written as:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = c_0\mathbf{1} + c_i\sigma_i \quad (1)$$

Having shown this you can pick and intelligent choice for the α_i in terms of the Pauli matrices, for example $\alpha_i = \sigma_i$ which automatically obeys $\{\alpha_i, \alpha_j\} = \mathbf{2}\delta_{ij}$, and express β in terms of $(\mathbf{1};\vec{\sigma})$ using (1) . Show then that there is no $\mathbf{2}\times\mathbf{2}$ β matrix that satisfies $\{\alpha_i, \beta\} = \mathbf{0}$.

Problem 2:

I. Show that the Weyl matrices:

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & \mathbf{0} \\ \mathbf{0} & \vec{\sigma} \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

satisfy all the Dirac conditions of Problem 1. Hence, they form just another representation of the Dirac matrices, the Weyl representation, which is different than the standard Pauli-Dirac representation.

II. Show the the Dirac matrices in the Weyl representation are

$$\vec{\gamma} = \begin{pmatrix} \mathbf{0} & \vec{\sigma} \\ -\vec{\sigma} & \mathbf{0} \end{pmatrix} \qquad \gamma^{\mathbf{0}} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

III. Show that in the Weyl representation $\gamma_5 = \boldsymbol{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$

IV. Solve the Dirac equation $[\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{p}}+\beta\,\boldsymbol{m}]\Psi = \boldsymbol{E}\,\Psi$ in the particle rest frame using the Weyl representation.

V. Compute the result of the chirality operators $\frac{(\mathbf{1}\pm\gamma_5)}{\mathbf{2}}$ when they are acting on the Dirac solutions in the Weyl representation.

Problem 3: Use the Dirac Hamiltonian in the standard Pauli-Dirac representation,

$$H = \vec{a}\cdot\vec{p}+\beta m$$

to compute $[H, \hat{L}]$ and $[H, \hat{\Sigma}]$ and show that they are not zero.
Use the results to show that:

$$[H, \hat{L}+(\frac{1}{2})\hat{\Sigma}] = 0$$

where the components of the angular momentum operator is given by:

$$\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$$

and the components of the spin operator are given by:

$$\hat{\Sigma}_i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

Recall that the Pauli matrices satisfy $\sigma^i\sigma^j = \delta^{ij}+i\epsilon^{ijk}\sigma^k$