Helicity and Chirality

Helicity: As we have seen before the helicity operator is defined as:

$$\vec{\Sigma}\cdot\hat{\boldsymbol{p}} = \begin{pmatrix} \vec{\sigma}\cdot\hat{\boldsymbol{p}} & \mathbf{0} \\ \mathbf{0} & \vec{\sigma}\cdot\hat{\boldsymbol{p}} \end{pmatrix} \qquad (1)$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the 2 x 2 Pauli matrices and $\hat{\boldsymbol{p}} = \vec{\boldsymbol{p}}/|\vec{\boldsymbol{p}}|$ is the unit vector at the direction of the momentum of a particle. As seen from (1) the **helicity represents the projection of the particle spin at the direction of motion**. It is easy to show that the helicity operator commutes with the Dirac hamiltonian:

$$[\vec{\Sigma}\cdot\hat{\boldsymbol{p}}, \boldsymbol{H}] = \mathbf{0} \qquad (2)$$

Hence, because of (2) the Dirac hamiltonian and helicity have a common set of eigenvectors. This is also the reason for the two-fold degeneracy found for every energy eigenstate of the Dirac hamiltonian. It is easy to show explicitly that the solutions of the Dirac equation are indeed eigenvectors of the helicity operator:

Consider the first two positive solutions of the Dirac Equation:

$$\Psi^{(1,2)}(\boldsymbol{x}) = \boldsymbol{N}\begin{pmatrix} \mathbf{1} \\ \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})} \end{pmatrix}\chi^{\pm}\,\mathrm{e}^{-ip^{\mu}x_{\mu}}$$

by applying the helicity operator we have:

$$(\vec{\Sigma}\cdot\hat{\boldsymbol{p}})\Psi^{(1,2)}(\boldsymbol{x}) = \boldsymbol{N}\begin{pmatrix} \vec{\sigma}\cdot\hat{\boldsymbol{p}} & \mathbf{0} \\ \mathbf{0} & \vec{\sigma}\cdot\hat{\boldsymbol{p}} \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})} \end{pmatrix}\chi^{\pm}\,\mathrm{e}^{-ip^{\mu}x_{\mu}} \Rightarrow$$

$$(\vec{\Sigma}\cdot\hat{\boldsymbol{p}})\Psi^{(1,2)}(\boldsymbol{x}) \;=\; \boldsymbol{N}\begin{pmatrix} \vec{\sigma}\cdot\hat{\boldsymbol{p}} \\ \vec{\sigma}\cdot\hat{\boldsymbol{p}}\,\dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})} \end{pmatrix}\chi^{\pm}\,\mathbf{e}^{-ip^{\mu}x_{\mu}} \;=\; \boldsymbol{N}\begin{pmatrix} \mathbf{1} \\ \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})} \end{pmatrix}\vec{\sigma}\cdot\hat{\boldsymbol{p}}\,\chi^{\pm}\,\mathbf{e}^{-ip^{\mu}x_{\mu}} \;\Rightarrow$$

$$(\vec{\Sigma}\cdot\hat{\boldsymbol{p}})\Psi^{(1,2)}(\boldsymbol{x}) \;=\; \boldsymbol{N}\begin{pmatrix} \mathbf{1} \\ \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})} \end{pmatrix}(\pm)\chi^{\pm}\,\mathbf{e}^{-ip^{\mu}x_{\mu}} \;=\; \pm\boldsymbol{N}\begin{pmatrix} \mathbf{1} \\ \dfrac{(\vec{\sigma}\cdot\vec{\boldsymbol{p}})}{(\boldsymbol{E}+\boldsymbol{m})} \end{pmatrix}\chi^{\pm}\,\mathbf{e}^{-ip^{\mu}x_{\mu}} \;\Rightarrow$$

$$(\vec{\Sigma}\cdot\hat{\boldsymbol{p}})\Psi^{(1,2)}(\boldsymbol{x}) \;=\; \pm\Psi^{(1,2)}(\boldsymbol{x})$$

Hence, we have shown that the eigenvectors of the Dirac hamiltonian are also eigenvectors of the helicity operator. In the last step we have used the relationship $\vec{\sigma}\cdot\hat{\boldsymbol{p}}\,\chi^{\pm} \;=\; \pm\chi^{\pm}$ which can easily be proven by selecting the unit vector at the direction of the z-axis.

It is important to notice that the Dirac solutions are eigenvectors of the helicity operator and in general not eigenvectors of the spin operator:

$$\vec{\Sigma} \;=\; \begin{pmatrix} \vec{\sigma} & \mathbf{0} \\ \mathbf{0} & \vec{\sigma} \end{pmatrix}$$

except in the case where the momentum is zero. Why should they be anyway ? The spin operator does not commute with the hamiltonian as we have seen before.

Therefore the helicity operator has the following properties:

(a) **Helicity is a good quantum number:** The helicity is conserved always because it commutes with the Hamiltonian. That is, its value does not change with time within a given reference frame. As we have seen before (2) is valid for both massive and massless fermions. Hence, helicity is conserved for both massive and massless particles.

(b) **Helicity is not Lorentz invariant:** This is obvious since helicity is a product of a 3-vector with an axial vector.

Chirality or Handedness:

Consider now the chirality/handedness operator in the Pauli-Dirac representation:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

which satisfies:

$$\{\gamma_{5,}\gamma^\mu\} = 0$$

This anti-commutation relationship is true in any Dirac matrix representation. Lets evaluate the commutator of the chirality operator with the Dirac hamiltonian:

$$[\gamma_5, H] = [\gamma_5, \vec{\alpha}\cdot\vec{p} + m\beta] = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} - \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \Rightarrow$$

$$[\gamma_5, H] = \begin{pmatrix} \vec{\sigma}\cdot\vec{p} & -m \\ m & \vec{\sigma}\cdot\vec{p} \end{pmatrix} - \begin{pmatrix} \vec{\sigma}\cdot\vec{p} & m \\ -m & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \Rightarrow$$

$$[\gamma_5, H] = 2m\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

So the chirality/handedness operator does not commute with the hamiltonian unless if the mass is zero. **Hence, although we don't know yet the physical observable which is associated with this operator we do know that it is conserved and corresponds to a good quantum number only if the mass is zero or can be neglected.**

Exercise 1: Consider the Dirac Hamiltonian for massless fermions $H = \vec{\alpha}\cdot\vec{p}$. Use the anti-commutation relationships $\{\gamma_{5,}\gamma^\mu\} = 0$ which are valid in any representation of the Dirac matrices to show that the result $[\gamma_5, H] = 0$ is true in any representation provided that the mass is zero.

Solution: $[H, \gamma_5] = [\gamma^0 \gamma^i p^i, \gamma_5] = p^i \{\gamma^0 [\gamma^i, \gamma_5] + [\gamma^0, \gamma_5] \gamma^i\} \Rightarrow$

$$[H, \gamma_5] = [\gamma^0 \gamma^i p^i, \gamma_5] = p^i \{\gamma^0 (\gamma^i \gamma_5 - \gamma_5 \gamma^i) + (\gamma^0 \gamma_5 - \gamma_5 \gamma^0) \gamma^i\} \Rightarrow$$

$$[H, \gamma_5] = [\gamma^0 \gamma^i p^i, \gamma_5] = p^i \{\gamma^0 (\gamma^i \gamma_5 + \gamma_5 \gamma^i) - (\gamma^0 \gamma_5 + \gamma_5 \gamma^0) \gamma^i\} \Rightarrow$$

$$[H, \gamma_5] = 0$$

If you try adding a mass term you get :

$$[H, \gamma_5] = [\vec{\alpha} \cdot \vec{p} + m \gamma^0, \gamma_5] = m(\gamma^0 \gamma_5 - \gamma_5 \gamma^0) = -2m \gamma_5 \gamma^0$$

Exercise 2: Show explicitly that for massless fermions the chirality and the Dirac hamiltonian have a common set of eigenfunctions which is expected because they commute.

Solution: This is easy to show: The eigenvectors of γ_5 are $\Psi^{\pm} = C\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$ with eigenvalues ± 1 respectively.

Consider the positive energy solutions of the Dirac Equation:

$$\Psi^{(1,2)}(x) = N \begin{pmatrix} 1 \\ \dfrac{(\vec{\sigma} \cdot \vec{p})}{(E+m)} \end{pmatrix} \chi^{\pm} e^{-ip^{\mu} x_{\mu}}$$

If the mass is zero we have that:

$$\Psi^{(1,2)}(x) = N \begin{pmatrix} 1 \\ (\vec{\sigma} \cdot \hat{p}) \end{pmatrix} \chi^{\pm} e^{-ip^{\mu} x_{\mu}} = N \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \chi^{\pm} e^{-ip^{\mu} x_{\mu}}$$

which is also an eigenfunction of the chirality operator.

Exercise 3: Show that for massless fermions if Ψ is a solution of the Dirac equation then $\gamma_5 \Psi$ is also a solution of the same equation.

Solution: The Dirac equation,

$$[i\gamma^{\mu}\partial_{\mu} - m]\Psi(x) = 0$$

for massless fermions becomes:

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = 0 .$$

Using $\{\gamma_{5,}\gamma^{\mu}\} = 0$ we get:

$$i\gamma^{\mu}\partial_{\mu}(\gamma_5 \Psi(x)) = 0$$

Lets now investigate the physical meaning of the chirality. Consider the massless Dirac equation:

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = 0$$

Let $\Psi(x) = u(\vec{p})\mathrm{e}^{-ip^{\mu}x_{\mu}}$ be a solution of the Dirac equation. By substituting we get that:

$$(\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p})u(\vec{p}) = 0 \Rightarrow$$

$$\gamma^0 p_0 u(\vec{p}) = \vec{\gamma}\cdot\vec{p}\, u(\vec{p}) \Rightarrow$$

$$\gamma_5\gamma^0\gamma^0 p_0 u(\vec{p}) = \gamma_5\gamma^0\vec{\gamma}\cdot\vec{p}\, u(\vec{p}) \Rightarrow$$

$$p_0\gamma_5 u(\vec{p}) = \gamma_5\gamma^0\vec{\gamma}\cdot\vec{p}\, u(\vec{p}) \qquad (3)$$

If this is a positive energy solution then we have that $p^0 > 0$ and (3) becomes:

$$\gamma_5 u(\vec{p}) = \gamma_5 \gamma^0 \vec{\gamma} \cdot \hat{p}\, u(\vec{p}) \qquad (4)$$

If this is a negative solution then, $p^0 < 0$ and

$$\gamma_5 u(\vec{p}) = -\gamma_5 \gamma^0 \vec{\gamma} \cdot \hat{p}\, u(\vec{p}) \qquad (5)$$

Lets compute the matrix product on the right side:

$$\gamma_5 \gamma^0 \vec{\gamma} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \vec{\Sigma} \Rightarrow$$

$$\vec{\Sigma} = \gamma_5 \gamma^0 \vec{\gamma} \qquad (6)$$

and **this is the definition of the spin operator** in terms of the gamma matrices valid in any representation. From (4) (5) and (6) we have that for:

$$p^0 > 0 \quad \Rightarrow \quad \gamma_5 u(\vec{p}) = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} u(\vec{p}) = \vec{\Sigma}\cdot\hat{p}\, u(\vec{p}) \qquad (7)$$

and

$$p^0 < 0 \quad \Rightarrow \quad \gamma_5 u(\vec{p}) = -\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} u(\vec{p}) = -\vec{\Sigma}\cdot\hat{p}\, u(\vec{p}) \qquad (8)$$

Using (7) and the fact that the Dirac spinors are eigenvectors of the helicity operator i.e.

$$[\vec{\Sigma}\cdot\hat{p}]\, u(\vec{p}) = \pm u(\vec{p})$$

we conclude that when acting on <u>positive energy solutions</u> the operators:

$$P_L = \frac{(1-\gamma_5)}{2} = \frac{(1-\vec{\Sigma}\cdot\hat{p})}{2}$$

and

$$P_R = \frac{(1+\gamma_5)}{2} = \frac{(1+\vec{\Sigma}\cdot\hat{p})}{2}$$

project to negative and positive helicity states respectively. Equivalently, using (8), when the above operators act on negative energy solutions they project to positive and negative helicity states respectively. Hence, we have the physical interpretation for the chirality operator: **The chirality or handedness is the same as the helicity operator when the particle mass is zero or it can be neglected.** The operators P_L and P_R are commonly referred as left handed and right handed projection operators.

Projection Operator Summary:

In general if Ψ^+ is a positive energy spinor and Ψ^- is a negative energy spinor we have that:

$$\frac{1+\gamma_5}{2}\Psi^{\pm} = +\frac{N}{2}\begin{pmatrix}1\\1\end{pmatrix}(1\pm\vec{\sigma}\cdot\hat{p})\chi^s$$

$$\frac{1-\gamma_5}{2}\Psi^{\pm} = \pm\frac{N}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1\mp\vec{\sigma}\cdot\hat{p})\chi^s$$

In the above relations we have changed the notation for the two dimensional spinors from $\chi^{\pm}$ to χ^s so that the $\pm$ spin sign is not confused with the the positive/negative energy $\pm$ sign. As before $\chi^s = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ for **s = 0,1**.

Conclusions on Chirality or Handedness:

For **massless** particles the chirality or handedness operator has the following properties:

(a) It is Lorentz invariant.
(b) It is conserved.
(c) It has a common set of eigenvectors with the Dirac Hamiltonian.
(d) It has the same properties with the Helicity operator. Which gives it a physical meaning.

Helicity and Chirality for massive particles:

So far we considered chirality/handedness for massless fermions. However, the chirality properties of massive fermions are also of interest. The reason for this is that the charged current weak interaction couples to left handed (negative chirality) particle spinors and right handed (postitive chirality) anti-particle spinors. Hence, we need a way to associate left handed and right handed spinors with positive an negative helicity states.

Consider the identity:

$$1-\frac{\vec{\sigma}\,\vec{p}}{E+M} = \frac{1}{2}(1-\frac{|\vec{p}|}{E+M})(1+\vec{\sigma}\cdot\hat{p})+\frac{1}{2}(1+\frac{|\vec{p}|}{E+M})(1-\vec{\sigma}\cdot\hat{p}) \qquad (9)$$

where $E\,,\ M\,,\ \vec{p}$ are the energy, mass and momentum of a fermion respectively.

Next consider a left handed operator acting on a positive Dirac solution:

$$\Psi_L = \frac{(1-\gamma_5)}{2}\Psi(x) = \frac{N}{2}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}\begin{pmatrix} 1 \\ \dfrac{(\vec{\sigma}\cdot\vec{p})}{(E+m)} \end{pmatrix}\chi^{\pm}\,\mathbf{e}^{-ip^{\mu}x_{\mu}} \Rightarrow$$

$$\Psi_L = \frac{N}{2}\begin{pmatrix} +1 \\ -1 \end{pmatrix}\left(1-\frac{(\vec{\sigma}\cdot\vec{p})}{(E+m)}\right)\chi^{\pm}\,\mathbf{e}^{-ip^{\mu}x_{\mu}}$$ and using (9) we get that:

$$\Psi_L = \frac{N}{2}\begin{pmatrix}+1\\-1\end{pmatrix}\left[\frac{1}{2}\left(1-\frac{|\vec{p}|}{E+M}\right)(1+\vec{\sigma}\cdot\hat{p})+\frac{1}{2}\left(1+\frac{|\vec{p}|}{E+M}\right)(1-\vec{\sigma}\cdot\hat{p})\right]\chi^{\pm}\,\mathrm{e}^{-ip^{\mu}x_{\mu}} \quad (10)$$

The first term in (10) projects to positive helicity states and the second term to negative helicity states. However, the coefficient of the positive helicity term vanishes at high energies where the particle mass can be neglected while the coefficient of the negative helicity term approaches the value of one at high energies.

One can show that at energies much larger than the particle mass these coefficients become:

$$\left(1-\frac{|\vec{p}|}{E+M}\right)=1-\frac{\sqrt{E^2-M^2}}{(E+M)}=1-\frac{\sqrt{1-M^2/E^2}}{(1+M/E)}=1-\left(1-\frac{M^2}{2E^2}+..\right)\left(1-\frac{M}{E}+..\right) \Rightarrow$$

$$\left(1-\frac{|\vec{p}|}{E+M}\right) \approx \frac{M}{E} \quad \text{to order of } \mathbf{M/E.}$$

and

$$\left(1+\frac{|\vec{p}|}{E+M}\right)=1+\frac{\sqrt{E^2-M^2}}{(E+M)}=1+\frac{\sqrt{1-M^2/E^2}}{(1+M/E)}=1+\left(1-\frac{M^2}{2E^2}+..\right)\left(1-\frac{M}{E}+..\right) \Rightarrow$$

$$\left(1+\frac{|\vec{p}|}{E+M}\right) \approx 2-\frac{M}{E} \quad \text{to order of } \mathbf{M/E.}$$

Hence,

$$\Psi_L \approx \frac{N}{2}\begin{pmatrix}+1\\-1\end{pmatrix}\left[\left(\frac{M}{2E}\right)(1+\vec{\sigma}\cdot\hat{p})+\left(1-\frac{M}{2E}\right)(1-\vec{\sigma}\cdot\hat{p})\right]\chi^{\pm}\,\mathrm{e}^{-ip^{\mu}x_{\mu}}$$

Therefore the left handed operator acting on positive energy states of the Dirac equation gives:

$$\Psi_L = \frac{(1-\gamma_5)}{2}\Psi(x) \approx \frac{N}{2}\begin{pmatrix}+1\\-1\end{pmatrix}[(\frac{M}{2E})(1+\vec{\sigma}\cdot\hat{p})+(1-\frac{M}{2E})(1-\vec{\sigma}\cdot\hat{p})]\chi^{\pm}\mathbf{e}^{-ip^{\mu}x_{\mu}}$$

As seen here the left handed positive energy spinor has contributions from both positive and negative helicity components. However the negative helicity component is dominant and becomes 100% in the case where the mass is much smaller than the energy and can be neglected. The positive helicity component decreases $\sim M/E$ and approaches zero as the energy increases.