The Dirac Equation under Lorentz and Parity Transformations

In the last lecture we studied the solutions of the Dirac equation which in covariant form is given by:

$$[i\gamma^{\mu}\partial_{\mu} - m]\Psi(x) = 0 \qquad (1)$$

The Dirac matrices obey the anti-commutation relationships:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \qquad (2)$$

However, we still have not proven that this equation is Lorentz invariant which is the basic requirement for all serious theories and equations in physics. We still don't know how a spinor transforms under Lorentz transformations. In fact we don't even know yet if a field $\Psi'(x')$ in the moving frame **O'** can be constructed in terms of the field $\Psi(x)$ in the rest frame **O** such that the Dirac equation remains invariant. It is tempting to say that $\gamma^{\mu}\partial_{\mu}$ must be a Lorentz invariant since it sure looks like the dot product of two 4-vectors. Hence, the answer must be trivial $\Psi'(x') = \Psi(x)$ just like it was with the Klein Gordon Equation. However tempting it might be, it is also wrong. Nowhere before have we shown that γ^{μ} is a contravariant 4-vector and in fact it is not. The approach we followed in the Klein Gordon equation falls apart. Hence, we suspect that $\Psi(x)$, γ^{μ} must transform in a more complicated way than the scalar field. In this lecture it will be shown that for a given Lorentz transformation there is a transformation of $\Psi(x)$, γ^{μ} which leaves the Dirac equation invariant.

The same is true for Parity transformations. Here too one cannot claim that γ^{μ} transforms like a polar vector under parity and follow the Klein-Gordon approach. The subject of this lecture is to show that the Dirac equation is in fact Lorentz and Parity invariant and to show how the spinor fields should transform under Parity and Lorentz transformations so that the Dirac equation remains invariant.

Lorentz Covariance of the Dirac Equation

Consider the infinitesimal Lorentz transformation, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$, which transforms tensors from reference frame **O** to the reference frame **O'**. The tensor $\omega^{\mu}{}_{\nu}$ generates the infinitesimal Lorentz transformation and it is considered to be a quantity that is smaller than one. In other words the major contribution to $\Lambda^{\mu}{}_{\nu}$ comes from the unity matrix $\delta^{\mu}{}_{\nu}$ and $\omega^{\mu}{}_{\nu}$ generates either an infinitesimal boost or an infinitesimal rotation or both. As shown before, 4-vectors in the reference frame **O** transform in to the **O'** reference frame as:

$$x^{\mu}{}' = \Lambda^{\mu}{}_{\nu} x^{\nu} = (\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}) x^{\nu} \quad \text{and}$$

$$x_{\mu}{}' x^{\mu}{}' = \Lambda_{\mu}{}^{\alpha} x_{\alpha} \Lambda^{\mu}{}_{\beta} x^{\beta} = (\delta_{\mu}{}^{\alpha} + \omega_{\mu}{}^{\alpha}) x_{\alpha} (\delta^{\mu}{}_{\beta} + \omega^{\mu}{}_{\beta}) x^{\beta} \Rightarrow$$

$$x_{\mu}{}' x^{\mu}{}' = (\delta_{\mu}{}^{\alpha} \delta^{\mu}{}_{\beta} + \delta_{\mu}{}^{\alpha} \omega^{\mu}{}_{\beta} + \omega_{\mu}{}^{\alpha} \delta^{\mu}{}_{\beta} + \omega_{\mu}{}^{\alpha} \omega^{\mu}{}_{\beta}) x_{\alpha} x^{\beta} \Rightarrow$$

and by neglecting terms which are quadratic in $\omega^{\alpha}{}_{\beta}$ we get that:

$$x_{\mu}{}' x^{\mu}{}' = (\delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} + \omega_{\beta}{}^{\alpha}) x_{\alpha} x^{\beta} = x_{\alpha} x^{\alpha} + (\omega^{\alpha}{}_{\beta} + \omega_{\beta}{}^{\alpha}) x_{\alpha} x^{\beta}$$

Since the dot product of two 4-vectors must be invariant under Lorentz it must be that

$$(\omega^{\alpha}{}_{\beta} + \omega_{\beta}{}^{\alpha}) = 0 \Rightarrow \omega_{\alpha\beta} = -\omega_{\beta\alpha}$$

Therefore the generator of the Lorentz transformations must be an antisymmetric matrix.

To prove that the Dirac equation is invariant under Lorentz transformations consider a spin half particle moving in space and an observer in a rest frame of reference, **O** , making measurements and determining that the properties of this fermion are described by the Dirac equation shown in (1). Consider a second observer in a moving frame, **O',** who also measures the properties of the same fermion in his frame. Both observers realize that they are related by a Lorentz transformation (for example they can see that the move closer or away from each other) and know how to translate the values of a given quantity from one reference frame to the other (they both know special relativity).

If the Dirac equation is going to be the same (invariant) for the two observers it must be that the observer in the **O'** coordinate frame concludes, by observing the fermion, that is satisfies the equation:

$$[i \tilde{\gamma}^{\mu} \partial_{\mu}{}' - m] \Psi'(x') = 0 \qquad (3)$$

The question is, if there is a choice of $\tilde{\gamma}^{\mu}$, $\Psi'(x')$ which satisfy (3) and can be derived from the corresponding quantities in the **O** reference frame. Lets start by assuming that a $\Psi'(x')$ exists such that:

$$\Psi'(x') = S(\Lambda)\Psi(x) \qquad (4)$$

It seems to be a reasonable assumption that the object that transforms the spinor $\Psi(x)$ from one frame to the other should be a 4 x 4 object which must depend upon the Lorentz transformation matrix Λ . Of course if (4) is true then it must be that:

$$\Psi(x) = S(\Lambda)^{-1}\Psi'(x') = S(\Lambda^{-1})\Psi'(x') \qquad (5)$$

$$(1)(5) \Rightarrow [i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m]S(\Lambda)^{-1}\Psi'(x') = 0 \Rightarrow$$

$$[i\gamma^{\mu}(\Lambda^{-1})_{\mu}^{\alpha}\frac{\partial}{\partial x^{\alpha}{'}} - m]S(\Lambda)^{-1}\Psi'(x') = 0 \Rightarrow \quad \text{(check this out)}$$

$$[iS(\Lambda)\gamma^{\mu}S(\Lambda)^{-1}(\Lambda^{-1})_{\mu}^{\alpha}\frac{\partial}{\partial x^{\alpha}{'}} - m]\Psi'(x') = 0 \qquad (6)$$

Hence, by comparing (6) with (3) we conclude that the matrix $S(\Lambda)$ must satisfy:

$$S(\Lambda)\gamma^{\mu}S(\Lambda)^{-1}(\Lambda^{-1})_{\mu}^{\alpha} = \gamma^{\alpha} \Rightarrow$$

$$\gamma^{\alpha} = (\Lambda)^{\alpha}_{\ \mu}S(\Lambda)\gamma^{\mu}S(\Lambda)^{-1} \qquad (7)$$

And the question is if such a $S(\Lambda)$ exists. It turns out that it does exists and has the form:

$$S(\Lambda) = \mathbf{e}^{\frac{-i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}} \qquad (8)$$

where:

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu} \quad \text{and} \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$$

The matrix $S(\Lambda)$ obeys: $\qquad S^{-1} = \gamma^{0}S^{+}\gamma^{0}$

Hence, from (6), (7), (8) we have that he Dirac equation is Lorentz invariant provided that the spinor transforms under (8) as:

$$\Psi'(x') = S(\Lambda)\Psi(x)$$

In other words the observer in **O'** will conclude that the physics of spin half fermions is also described in his reference frame by the Dirac equation. However, his coordinate and time measurements will be different (in terms of $x^{\mu}{}'$) and his spinor will also be different: $S(\Lambda)\Psi(x) = S(\Lambda)\Psi(\Lambda^{-1}x')$. So as seen here the spinor field transforms under Lorentz in a very different way than the scalar field.

The Dirac Equation under Parity Transformations

In a similar way one can study the properties of the Dirac equation under parity. Unlike the Lorentz transformation parity is a discrete transformations where:

$$P: \quad t \rightarrow t' = t \quad ; \quad \vec{x} \rightarrow \vec{x}' = -\vec{x} \qquad \text{(P)}$$

As before we assume that we have a spin half particle and we are interested to study its properties as seen from to reference frames: **O** and it its parity inverted partner **O'.** Consider also two observers, one in each frame, that know that each of them lives in a parity inverted world. Do both conclude that the particle obeys the Dirac equation ?

Start with the Dirac equation in the **O** frame:

$$[i\gamma^{\mu}\partial_{\mu} - m]\Psi(x) = 0 \Rightarrow$$

$$[i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m]\Psi(x) = 0 \Rightarrow$$

$$[i\gamma^{0}\frac{\partial}{\partial x^{0}} + i\vec{\gamma}\vec{\nabla}_{\vec{x}} - m]\Psi(x) = 0 \qquad (1)$$

where $\vec{\nabla}_{\vec{x}}$ is the nabla operator as a function of the vector $\vec{x}$. Note that the sign for the vector product is plus and it is correct because of the definition of the 4-derivative:

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = (\frac{\partial}{\partial x^{0}}; +\vec{\nabla})$$

So equation (1) has the time and the space variable separate and the observer in **O** can use it to predict how would it look in the parity inverted world of **O'**:

$$[i\gamma^0 \frac{\partial}{\partial x^0} - i\vec{\gamma}\vec{\nabla}_{\vec{x}}' - m]\Psi(-\vec{x}', t) = 0 \quad (2)$$

where $\vec{\nabla}_{\vec{x}}'$ is the nabla operator as a function of the vector $\vec{x}'$. From the first point of view this is not the Dirac equation precisely because of the minus in the vector product which came up when we translated ∇ in the parity inverted system. However this can be repaired by multiplying both sides by γ^0 and using the Dirac matrices anti-commutation relationships $\{\gamma^0, \gamma^i\} = 2g^{0i} = 0$:

From (2) we have that: $\gamma^0[i\gamma^0 \frac{\partial}{\partial x^0} - i\vec{\gamma}\vec{\nabla}_{\vec{x}}' - m]\Psi(-\vec{x}', t) = 0 \Rightarrow$

$$[i\gamma^0 \frac{\partial}{\partial x^0} + i\vec{\gamma}\vec{\nabla}_{\vec{x}}' - m]\gamma^0\Psi(-\vec{x}', t) = 0 \quad (3)$$

Equation (3) is the Dirac equation and if the free spin-half fermion physics is to be parity invariant the observer in **O'** should conclude that the fermion in his frame obeys:

$$[i\gamma^0 \frac{\partial}{\partial x^0} + i\vec{\gamma}\vec{\nabla}_{\vec{x}}' - m]\Psi'(\vec{x}', t) = 0 \quad (4)$$

Hence, the Dirac equation is invariant under parity if at the same time that we change the coordinates according to (P) we also transform the spinor as:

$$\Psi'(\vec{x}', t) = e^{i\phi}\gamma^0\Psi(\vec{x}, t) = e^{i\phi}\gamma^0\Psi(-\vec{x}', t)$$

As seen the parity transformation is just a matrix multiplication by γ^0 up to an additional arbitrary phase.

Next we investigate what happens if we apply the parity operator on the solutions of the Dirac equation. Recall the solutions of the Dirac equation for massive spin-half fermions at the particle rest frame. The positive solutions of the Dirac equation were:

$$\Psi^1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \quad \text{and} \quad \Psi^2(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}$$

and the negative energy ones were:

$$\Psi^3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt} \quad \text{and} \quad \Psi^4(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

The effect of the parity operator on them is:

$$P(\Psi^1(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} = (+1)\Psi^1(t)$$

$$P(\Psi^2(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} = (+1)\Psi^2(t)$$

$$P(\Psi^3(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-imt} = (-1)\Psi^3(t)$$

$$P(\Psi^4(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} e^{-imt} = (-1)\Psi^4(t)$$

Hence, the negative energy solutions have opposite parity than the positive energy solutions. **As we shall see later the negative energy solutions are associated with antiparticles and the positive energy solutions with particles. Hence, anti-particles have opposite parity than particles.**

The effect of the parity operator on the general free-particle Dirac solution can be seen when it is acting on the positive energy Dirac solutions:

$$\Psi^{(1,2)}(x) = N\begin{pmatrix} 1 \\ \dfrac{(\vec{\sigma}\cdot\vec{p})}{(E+m)} \end{pmatrix} \chi^{\pm}\, e^{-ip^{\mu}x_{\mu}} = N\begin{pmatrix} 1 \\ \dfrac{(\vec{\sigma}\cdot\vec{p})}{(E+m)} \end{pmatrix} \chi^{\pm}\, e^{-ip^{0}x_{0}+i\vec{p}\cdot\vec{x}}$$

Hence,

$$\Psi^{(1,2)}{}'(\vec{x}\,',t) = P(\Psi^{(1,2)}(\vec{x},t)) = N\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 1 \\ \dfrac{(\vec{\sigma}\cdot\vec{p})}{(E+m)} \end{pmatrix} \chi^{\pm}\, e^{-ip^{0}x_{0}+i\vec{p}\cdot\vec{x}} \Rightarrow$$

$$\Psi^{(1,2)}{}'(\vec{x}\,',t) = = N\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot(-\vec{p})}{(E+m)} \end{pmatrix} \chi^{\pm}\, e^{-ip^{0}x_{0}-i\vec{p}\cdot\vec{x}\,'} \Rightarrow$$

$$\Psi^{(1,2)}{}'(\vec{x}\,',t) = = N\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot(-\vec{p})}{(E+m)} \end{pmatrix} \chi^{\pm}\, e^{-ip^{0}x_{0}+i(-\vec{p})\cdot\vec{x}\,'} = \Psi(\vec{x}\,',t,-\vec{p})$$

As expected the 3-momentum vector flips without any further change to the spinor form.

The parity that we have been discussed here is the **intrinsic parity** associated with each particle and this is in addition to the parity associated with its orbital wave function. In other words **the parity of the wave function of a particle or a system of particles is the product of the orbital parity times the intrinsic parity.**

In interactions where parity is conserved (electromagnetic and strong) we can use this to compute selection rules for the various reactions. We do this aided by the following two ideas:

- Under parity the orbital wave-function will change by $(-1)^L$ where L is the angular momentum quantum number. This is because a parity operation $\vec{r} \rightarrow \vec{r}\,' = -\vec{r}$ changes $\theta \rightarrow \theta - \pi \;\; ; \;\; \phi \rightarrow \phi + \pi$ and the orbital wave function involves spherical harmonics (central potential) which change as $(-1)^L$. This is actually the reason that the photon has negative parity, 1^-. Photons are emitted via atomic dipole transitions where $\Delta l = \pm 1$. Hence the atomic parity changes by (-1) during this transitions and for the overall parity of the system (atom + photon) to be conserved (electromagnetic interaction) then we must have that the photon has negative parity.

- The overall wave function must be symmetric for systems of bosons and antisymmetric for systems of fermions.

Example: The parity of the negative pion

The parity of the negative pion can determined using the capture reaction of a π^- from a deuterium. The capture takes place in the S-state and the spins of the deuterium and pion are $S_d = 1$ and $S_d = 0$ respectively. The deuterium has positive intrinsic parity. The capture reaction gives to neutrons (fermions) in the final state.

$$\pi^- + d \rightarrow n + n$$

The problem can be solved by noticing that:

(1) The total initial angular momentum is $J = 1$ because the orbital angular momentum is zero (S-state). Hence, it is only the deuterium spin which contributes.

(2) The complete wave function of the two neutrons in the final state is can be written as:

$$\Psi = \Phi(orbital) \times S(spin)$$

and must be antisymmetric under exchange of the two identical fermions.

(3) The spin wave function of the two final state fermions is given by:

$$|1,1> \; = \; |\uparrow\uparrow>$$

$$|1,1> \; = \; (\frac{1}{\sqrt{2}})(|\downarrow\uparrow> + |\uparrow\downarrow>) \qquad \text{(S=1; triplet)}$$

$$|1,-1> \; = \; |\downarrow\downarrow>$$

and

$$|0,0> \; = \; (\frac{1}{\sqrt{2}})(|\downarrow\uparrow> - |\uparrow\downarrow>) \qquad \text{(S=0; singlet)}$$

From (2) and (3) we have that the parity of the neutron wave function will be:

$$(-1)^{L+S+1}$$

Note that: (a) in the two-body wave function the interchange of the two fermions amounts to parity inversion. (b) the singlet is antisymmetric and and triplet is symmetric. Since the total wave function must be anti-symmetric it means that:

$$(-1)^{L+S+1} \; = \; -1 \; \Rightarrow \; L+S \; = \; even$$

Hence, there are several possibilities:

L	S	J	L+S odd/even	OK/NOT OK
0	1	1	odd	Not OK
1	0	1	odd	Not OK
1	1	0,1,2	even	OK
2	1	1,2,3	odd	Not OK

And only the third one has both total angular momentum equal to one and **L+S** even.

Conclusion:

This means that the parity of the neutron system is: $(-1)^{L} = (-1)^{1} = -1$ since the intrinsic neutron parity appears in square.

Because the pion capture is an electromagnetic reaction it conserves parity. Hence the parity of the pion deuterium system must also be **-1**. Considering that this system has angular momentum zero and that the parity of the deuterium is +1 we conclude that the pion must have negative parity. **Hence the negative pion is a pseudoscalar or 0^- .**

Bilinear Covariants:

Now that the Lorentz and parity transformations of the spinors are known we can construct quantities which have definite Lorentz and parity properties:

Example 1: Consider the spinor function: $\bar{\Psi}(x)\Psi(x)$. It can be shown that this is a scalar. That is invariant under Lorentz and with positive parity.

Solution: Consider this quantity in the **O'** reference frame and study how does it transform under Lorentz transformations:

$$\bar{\Psi}'(x')\Psi'(x') = \Psi^{+\prime}(x')\gamma^{0}\Psi'(x') = (S(\Lambda)\Psi(x))^{+}\gamma^{0}S(\Lambda)\Psi(x) \Rightarrow$$

$$\bar{\Psi}'(x')\Psi'(x') = \Psi(x)^{+}S(\Lambda)^{+}\gamma^{0}S(\Lambda)\Psi(x) \Rightarrow$$

$$\bar{\Psi}'(x')\Psi'(x') = \Psi(x)^{+}\gamma^{0}\gamma^{0}S(\Lambda)^{+}\gamma^{0}S(\Lambda)\Psi(x) \Rightarrow$$

$$\bar{\Psi}'(x')\Psi'(x') = \bar{\Psi}(x)\gamma^{0}S(\Lambda)^{+}\gamma^{0}S(\Lambda)\Psi(x) = \bar{\Psi}(x)S^{-1}(\Lambda)S(\Lambda)\Psi(x) \Rightarrow$$

$$\bar{\Psi}'(x')\Psi'(x') = \bar{\Psi}(x)\Psi(x)$$. Hence, it is invariant under Lorentz.

Here we have used the property of Lorentz transformations: $S^{-1} = \gamma^{0}S^{+}\gamma^{0}$

Under parity it transforms as follows:

$$\bar{\Psi}'(\vec{x}',t)\Psi'(\vec{x}',t) = \Psi^{+\prime}(\vec{x}',t)\gamma^{0}\Psi'(\vec{x}',t) = (\gamma^{0}\Psi(\vec{x},t))^{+}\gamma^{0}\gamma^{0}\Psi(\vec{x},t) \Rightarrow$$

$$\bar{\Psi}'(\vec{x}',t)\Psi'(\vec{x}',t) = \Psi^{+}(\vec{x},t)\gamma^{0}\gamma^{0}\gamma^{0}\Psi(\vec{x},t) = \bar{\Psi}(\vec{x},t)\Psi(\vec{x},t) \quad .$$

Hence it is invariant under parity and Lorentz so it is a scalar.

Example 2: Consider the spinor function: $J^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x)$. It can be shown that this transforms as a polar vector under Lorentz and parity.

Solution:

Consider the transformation properties under Lorentz:

$$J^{\mu}{}'(x') = \bar{\Psi}'(x')\gamma^{\mu}\Psi'(x') = \Psi^{+}{}'(x')\gamma^{0}\gamma^{\mu}\Psi'(x') \Rightarrow$$

$$J^{\mu}{}'(x') = \Psi^{+}{}'(x')\gamma^{0}\gamma^{\mu}\Psi'(x') = \Psi'^{+}(x)S^{+}(\Lambda)\gamma^{0}\gamma^{\mu}S(\Lambda)\Psi(x) \Rightarrow$$

$$J^{\mu}{}'(x') = \Psi'^{+}(x)\gamma^{0}\gamma^{0}S^{+}(\Lambda)\gamma^{0}\gamma^{\mu}S(\Lambda)\Psi(x) \Rightarrow$$

$$J^{\mu}{}'(x') = \bar{\Psi}(x)\gamma^{0}S^{+}(\Lambda)\gamma^{0}\gamma^{\mu}S(\Lambda)\Psi(x) \qquad \text{(A)}$$

However we have shown that for any Lorentz spinor transformation we have:

$$S^{-1} = \gamma^{0}S^{+}\gamma^{0} \qquad \text{(B)}$$

and (A) and (B) give:

$$J^{\mu}{}'(x') = \bar{\Psi}(x)S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda)\Psi(x) \qquad \text{(C)}$$

However, as seen in (7) necessary condition for the Dirac Equation to be Lorentz invariant was:

$$\gamma^{\mu} = (\Lambda)^{\mu}{}_{\alpha}S(\Lambda)\gamma^{\alpha}S(\Lambda)^{-1} \Rightarrow$$

$$S(\Lambda)^{-1}\gamma^{\mu}S(\Lambda) = (\Lambda)^{\mu}{}_{\alpha}\gamma^{\alpha} \Rightarrow \qquad \text{(D)}$$

Using (C) and (D) we get that.

$$J^{\mu}{}'(x') = \bar{\Psi}(x)(\Lambda)^{\mu}{}_{\alpha}\gamma^{\alpha}\Psi(x) \Rightarrow$$

$$J^{\mu}{}'(x') = (\Lambda)^{\mu}{}_{\alpha}\bar{\Psi}(x)\gamma^{\alpha}\Psi(x)$$

Hence, $J^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x)$ transforms as a Lorentz vector.

Next see how does $J^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x)$ transforms under parity.

$$\bar{\Psi}'(\vec{x}',t)\gamma^{\mu}\Psi'(\vec{x}',t) = \Psi^{+}(\vec{x},t)\gamma^{0}\gamma^{0}\gamma^{\mu}\gamma^{0}\Psi(\vec{x},t) = \bar{\Psi}(\vec{x},t)\gamma^{0}\gamma^{\mu}\gamma^{0}\Psi(\vec{x},t)$$

Using the antic-commutation relationship $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}$ we can change the order that the gamma matrices are being multiplied:

$$\mu,\nu = 0: \qquad \gamma^{0}\gamma^{0}+\gamma^{0}\gamma^{0} = 2g^{00} = 2 \Rightarrow \gamma^{0}\gamma^{0} = 1$$

$$\mu=0,\nu=i=1,2,3: \qquad \gamma^{0}\gamma^{i}+\gamma^{i}\gamma^{0} = 2g^{0i} = 0 \Rightarrow \gamma^{0}\gamma^{i} = -\gamma^{i}\gamma^{0}$$

Hence, under parity we have that:

$$J^{0}{}'(x') = \bar{\Psi}'(x')\gamma^{0}\Psi'(x') = \Psi(x)\gamma^{0}\Psi(x) = J^{0}(x)$$

and

$$\vec{J}'(x') = \bar{\Psi}'(x')\vec{\gamma}\Psi'(x') = -\bar{\Psi}(x)\vec{\gamma}\Psi(x) = -\vec{J}(x)$$

Therefore $J^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x)$ transforms as a 4-vector under Lorentz and as a vector under parity hence it is a **Vector.** Sometime it is referred to as a **Polar Vector** to be distinguished from **Axial Vectors**.

In general one can show that the following quantities, called Bilinear Covariants, transform under Lorentz and Parity as follows:

(1) $\bar{\Psi}\Psi$ is a **Scalar (S).**

(2) $\bar{\Psi}\gamma^{5}\Psi$ is a **Pseudoscalar (P)**.

(3) $\bar{\Psi}\gamma^{\mu}\Psi$ is a **Polar Vector (V)**.

(4) $\bar{\Psi}\gamma^{\mu}\gamma^{5}\Psi$ is an **Axial Vector (A)**.

(5) $\bar{\Psi}[\gamma^{\mu},\gamma^{\nu}]\Psi$ is a **Tensor (T)**.

These quantities form a basis for expressing any 4x4 spinor current. Hence, one can use them to write down the most general from of a current for a given interaction.