APP - Lecture 10 - QED Cross Sections

10.1 Introduction

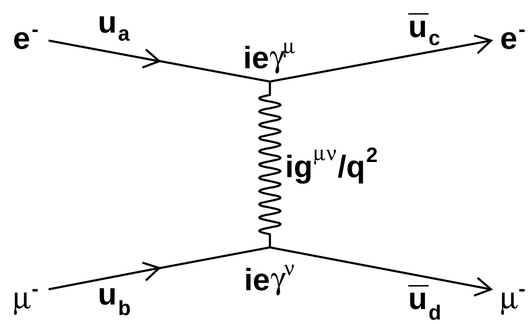
We have Fermi's Golden Rule and an expression for the phase space, so if we use the Feynman rules to calculate a matrix element, we can actually find a measurable quantity and compare it to data. We will look at two-to-two reactions today, where the cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 s}$$

The two we will consider are $e^-\mu^- \to e^-\mu^-$ and $e^+e^- \to \mu^+\mu^-$. The full calculation is lengthy; we will not do every step here, but will sketch out how it fits together.

10.2 The reaction $e^-\mu^- \to e^-\mu^-$

The Feynman diagram for this reaction is



The Feynman rules say that the matrix element is

$$iM = \overline{u}_c(ie\gamma^\mu)u_a \left(\frac{-ig^{\mu\nu}}{q^2}\right) \overline{u}_d(ie\gamma^\nu)u_b$$

Note the structure of the Dirac matrices is

$$\overline{u}\gamma u \sim \begin{pmatrix} . & . & . & . \end{pmatrix} \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \begin{pmatrix} . \\ . \\ . \\ . \end{pmatrix}$$

for each of the $\overline{u}\gamma u$ terms and so is simply a number. Hence, we can think in terms of "transition currents"

$$J_e^\mu = e\overline{u}_c\gamma^\mu u_a, \qquad J_m^\mu = e\overline{u}_d\gamma^\mu u_b$$

which are normal vectors (i.e. not matrices) and the matrix element becomes

$$M = \frac{J_e^\mu J_{m\mu}}{q^2}$$

These transition currents being separate for the electrons and the muons reflect the fact that they do not interact with each other directly, but only through a boson field, here the photon.

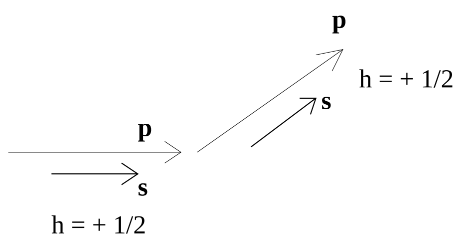
Hence, the idea is to plug in our solutions which we found in a previous lecture, calculate the transition currents and so find the matrix element. There is one complication, which is the spin. We found there are actually two possible u_a, corresponding to the two basis states ϕ_1 and ϕ_2 in the solutions. The same is true for the other u states also. Hence, there are actually $2 \times 2 \times 2 \times 2 = 16$ different combinations (and hence matrix elements) to calculate. Do they interfere, i.e. do we add the matrix elements and then square for the cross section, or square and then add? They are in principle distinguishable as they correspond to different polarisations of the incoming and outgoing particles, so we can square each of the 16 matrix elements separately.

In fact, in almost all experiments, the incoming beams are unpolarised, meaning an equal mixture of the two states, and outgoing particles do not have their polarisation measured. Therefore, in almost all cases of interest, we average over the four incoming combinations and sum over the outgoing four, so

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{i=1}^{16} |M_i|^2$$

As we are no longer interested in the spin states, we could use any basis for ϕ_1 and ϕ_2 and we would get the same answer for $\langle |M|^2 \rangle$. However, the helicity basis, where the spin is well defined and is aligned either parallel ($h = +1/2$) or antiparallel ($h = -1/2$), brings out some important physics and also reduces the number of contributions substantially.

Consider the transition current for the electrons when both have $h = +1/2$.



In this case, we can plug in the helicity solutions and the γ matrices and so find the four components of the transition current for this combination. However, for the case where the outgoing electron becomes $h = -1/2$, we find the transition current is zero. This is not specific to this reaction but is very general; the form of the current $\overline{\psi}\gamma^\mu\psi$ does not allow helicity to change so the helicity of a particle is conserved in electromagnetic interactions. (Note this is only exactly true in this high energy limit where we are working; we will find the exact conserved quantity when we consider the weak interactions.) Similarly, for the incoming electron having $h = -1/2$ and the outgoing $h = +1/2$, then the transition current is also zero while the fourth combination $h = -1/2$ and $h = -1/2$, gives a non-zero contribution.

Hence, there are only two non-zero transitions currents for the electron if we use the helicity basis. Clearly, the same is true for the muon, so there are now only $2 \times 2 = 4$, rather than 16, combinations to calculate. If we consider the transition current products, then when both the electron and the muon are $h = +1/2$, then the product is

$$J^\mu_{e+} J_{m+\mu} = -2ie^2 s$$

where s is the square of the centre-of-mass energy. The same is true for both of them negative also. In this case, the two spins are opposite so there is no preferred direction in space; hence we obtain a contribution which does not depend on θ, the angle of the outgoing electron compared with the incoming electron direction.

However, for the electron $h = +1/2$ and the muon $h = -1/2$, then the product gives

$$J^{\mu}_{e+} J_{m-\mu} = -2ie^2 s \cos^2(\theta/2)$$

Here the spins are parallel and do pick out a direction in space, allowing there to be a θ dependence. Again, reversing both spins gives the same answer.

The final thing we need is the photon $q^2 = E_\gamma^2 - p_\gamma^2$. Energy and momentum are conserved at each vertex, so the photon must have the difference between the incoming and outgoing electrons

$$p^{\mu}_{\gamma} = p^{\mu}_{a} - p^{\mu}_{c} = (E, 0, 0, E) - (E, E\sin\theta, 0, E\cos\theta) = [0, -E\sin\theta, 0, E(1-\cos\theta)]$$

so the photon has momentum, but no energy. The q^2 propagator factor is then

$$q^2 = p^{\mu}_{\gamma} p_{\gamma\mu} = -E^2\sin^2\theta - E^2(1-\cos\theta)^2 = -2E^2(1-\cos\theta) = -s\sin^2(\theta/2)$$

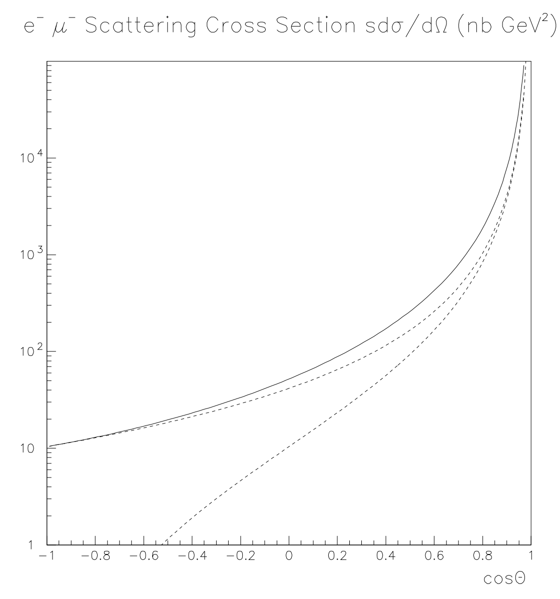
and so is negative. Putting all this together, the total matrix element is therefore

$$\langle |M|^2 \rangle = \frac{1}{4}\frac{8e^4s^2 + 8e^4s^2\cos^4(\theta/2)}{s^2\sin^4(\theta/2)} = 2e^4\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}$$

Using the result from the previous lecture for the cross section, then

$$\frac{d\sigma}{d\Omega} = \frac{\langle |M|^2 \rangle}{64\pi^2 s} = \frac{e^4}{32\pi^2 s}\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} = \frac{\alpha^2}{2s}\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}$$

using $\alpha = e^2/4\pi$. Hence, we have the cross section as a function of angle of the scattered electron.

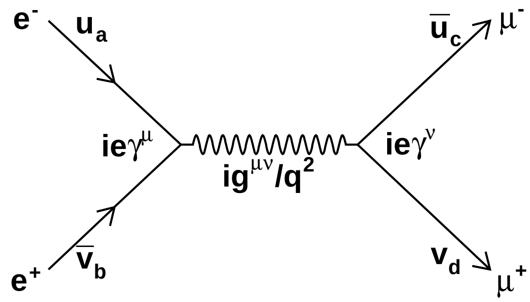


The very rapid rise of the cross section in the forward direction says the angle of scatter is usually very small. This is because the virtual photon then carries a small amount of momentum and so is closest to its "natural" mass of zero.

It turns out this cross section has not been thoroughly measured because of experimental difficulties with muon beams. This is not true of the second reaction, which has copious experimental data.

10.3 The reaction $e^+e^- \to \mu^+\mu^-$

The Feynman diagram for this reaction is



The calculation is very similar. We need to use the v rather than the u solutions for the positron and antimuon, but otherwise it looks the same. There are again 16 combinations but in the helicity basis, several of the transitions current are zero. In this case, it is the combinations where both the electron and positron (or muon and antimuon) have the same helicity which give zero. There are again four non-zero combinations, with two giving $ie^2 s(1-\cos\theta)$ and two giving $ie^2 s(1+\cos\theta)$.

The photon four-momentum is now the sum of the incoming particles

$$p_\gamma^\mu = p_a^\mu + p_b^\mu = (E,0,0,E) + (E,0,0,-E) = (2E,0,0,0)$$

In this case, the photon has energy but no momentum. Finding q^2 is easy

$$q^2 = 4E^2 = s$$

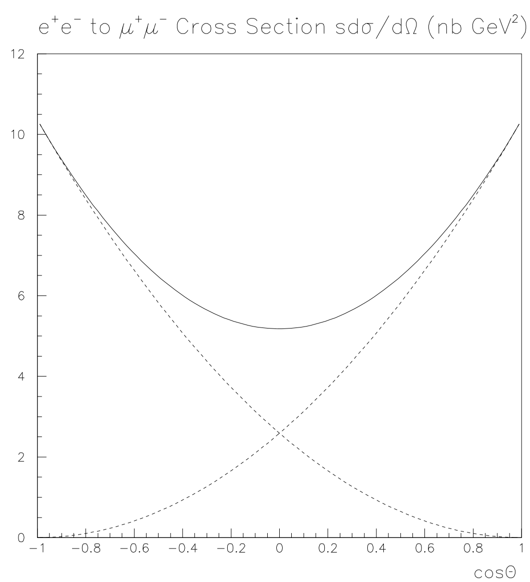
as it must have all the energy available in the middle of the diagram. The total is therefore

$$\langle |M|^2 \rangle = \frac{1}{4}\frac{2e^4 s^2(1-\cos\theta)^2 + 2e^4 s^2 (1+\cos\theta)^2}{s^2} = e^4(1+\cos^2\theta)$$

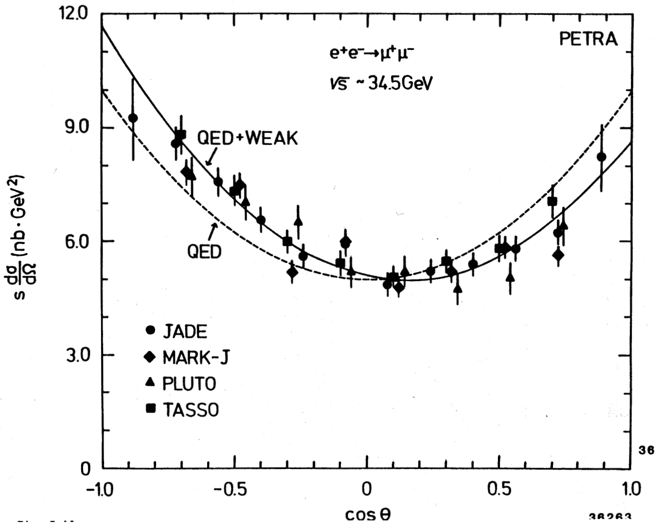
Using the result from the previous lecture for the cross section, then

$$\frac{d\sigma}{d\Omega} = \frac{\langle |M|^2\rangle}{64\pi^2 s} = \frac{e^4}{64\pi^2 s}(1+\cos^2\theta) = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

This looks like



and this has been measured both for muons



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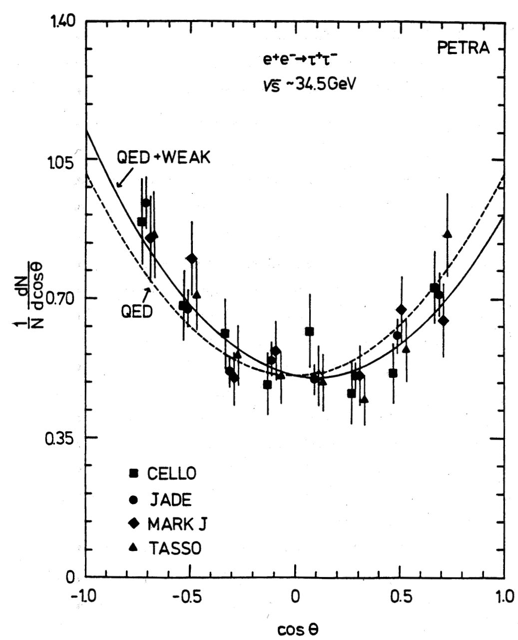
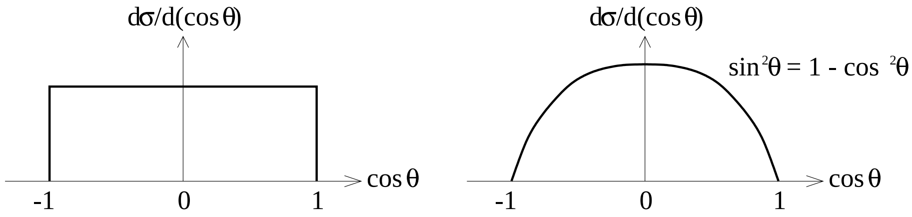


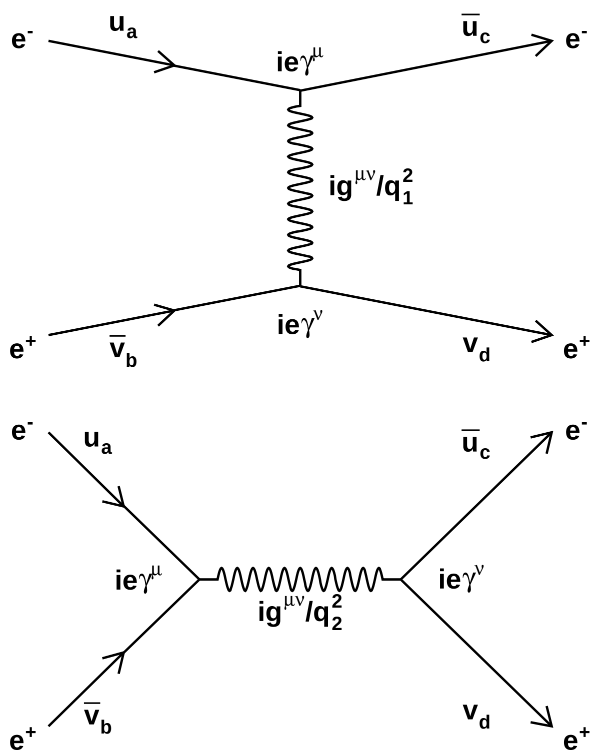
This is a significant result. There is good agreement; the shape and magnitude are clearly dominated by the photon exchange calculated here. The small discrepancy is understood in terms of weak effects not included here. Is this shape actually dependent on our assumptions? We are sensitive to the spin of both the photon and the muon here. If the photon was spin 0, there would be no preferred direction for the outgoing muons so a flat distribution in $\cos\theta$ would be expected. Alternatively, having spin 0 muons would give a $\sin^2\theta$ distribution.

Finally, note the $1/s$ behaviour; the cross section gets very small at high energies as the virtual photon is further off resonance; its mass is far from the natural mass of zero. Conversely, the cross section gets very large as the energy is reduced; hence at rest matter and antimatter annihilate very easily.

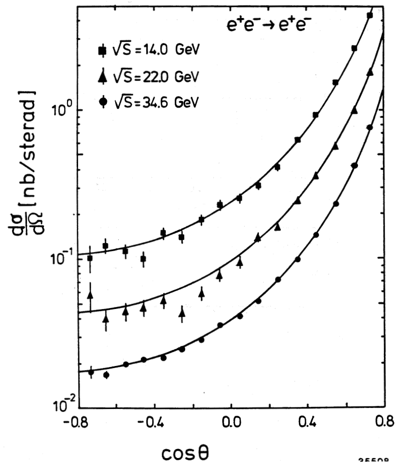


10.4 The reaction $e^+e^- \to e^+e^-$

You may be wondering why we have not looked at the reaction $e^+e^- \to e^+e^-$. This reaction is actually more complicated because there are two diagrams in this case; one for scattering like in $e^-\mu^- \to e^-\mu^-$ and one for annihilation followed by pair production as in $e^+e^- \to \mu^+\mu^-$. The Feynman diagrams for this reaction are

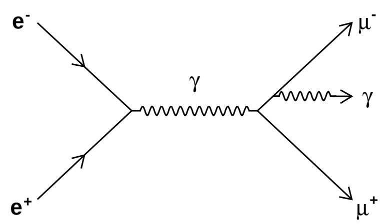


and now we must interfere them before taking the matrix element square as they are indistinguishable, even in principle. The result is similar to $e^-\mu^- \to e^-\mu^-$ as this diagram gives a much bigger contribution than the other.

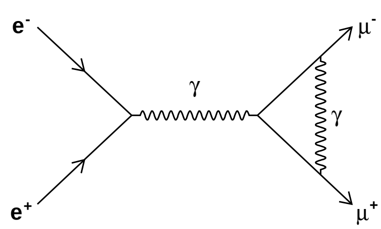


10.5 Higher order effects

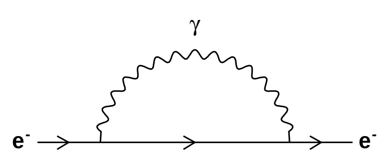
That is really it for QED to lowest order, i.e. $e^4 = \alpha^2 = 1/(137)^2$. There are higher order diagrams, such as



which has an extra vertex and so goes as $e^6 = \alpha^3 = 1/(137)^3$. They are reasonably straightforward to calculate and will then be $\mathcal{O}(\%)$ corrections. However, there are nastier diagrams such as



which looks α^4 but can interfere with the lowest order diagram and so contributes at $e^2 \times e^4 = e^6 = \alpha^3$ also. There are even diagrams like

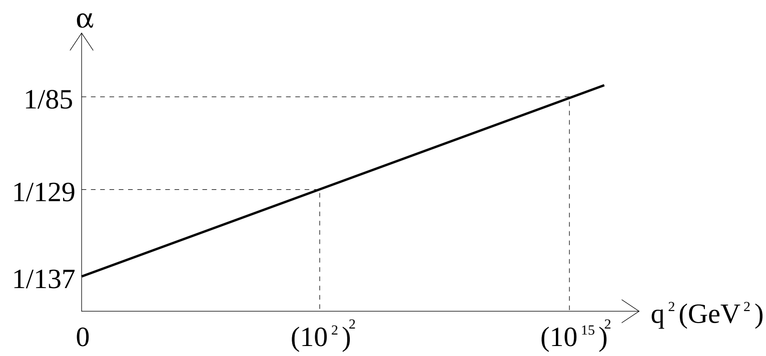


The critical point here is both these have a closed loop. It turns out energy and momentum conservation are not sufficient to define the momenta of particles within such a loop. Hence, we need to add the contributions for *any* momentum within the loop and this gives an infinite result. However, it turns out all such diagrams give the "same" infinity plus a finite term. Hence, differences between them, which are all we are sensitive to, are finite. This includes the mass and wavefunction normalisation and the latter is why this dodgy process is called "renormalisation".

One other such quantity is e; the "bare" coupling constant α also becomes infinite but the "observable" finite term is not so we get sensible answers. However, the observable value is not actually a constant but depends on the energy of the reaction, so $\alpha = \alpha(q^2)$; the fine structure constant is no longer constant. Obviously, the value obtained from atomic physics corresponds to the $q^2 = 0$ limit. In reality it depends very weakly on q^2 so

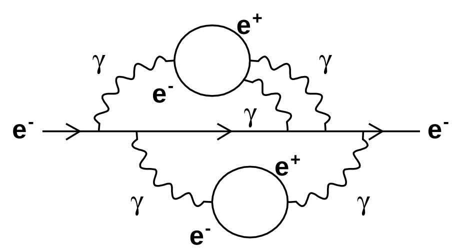
$$\alpha(0) \approx \frac{1}{137}, \qquad \alpha((10^2)^2\mathrm{GeV}^2) \sim \frac{1}{129}, \qquad \alpha((10^{15})^2\mathrm{GeV}^2) \sim \frac{1}{85}$$

or graphically



For all but the highest energy experiments, it is constant to a good approximation.

Physically, what is going on? The diagrams such as



mean a "free" electron is surrounded by a cloud of virtual photons, electrons and positrons. These latter screen the charge in a similar way to a dielectric, so in electrostatics we use $\boldsymbol{D}$ rather than $\boldsymbol{E}$. This is called "vacuum screening". A very high energy photon has a small enough de Broglie wavelength to resolve the "bare" electron but low energy photons only see the whole cloud and so some effective reduced charge. The *true* value of α is that seen at very high energies, not what we normally measure.