APP - Lecture 9 - Measurable rates

9.1 Introduction

We have the QED Lagrangian density and we now want to calculate some measurable quantity such as a cross section or decay rate. How do we connect the two? We need to put several pieces together and they are encapsulated in Fermi's Golden Rule.

9.2 Fermi's Golden Rule

The most basic equation we use is Fermi's Golden Rule

$$\text{Rate} = |M|^2 \rho \prod_{in} \frac{1}{2E_{in}}$$

where

- $|M|$ is the "matrix element" or "amplitude" and depends on the Lagrangian density,
- ρ is the "relativistic density of final states" (RDFS) or "Lorentz invariant phase space" (LIPS) and depends only on the final state,
- $1/E_i$ is a factor to compensate for the incoming wavefunction Lorentz normalisation and depends only on the initial state.

Fermi's Golden Rule gives us the rate of a process and there are two things we use it for; decays and reactions.

For a particle X decaying, then Fermi's Golden Rule tells us

$$\text{Rate} = |M|^2 \rho \frac{1}{2E_X}$$

This is not Lorentz invariant due to the $1/E_X$; specifically the higher the particle energy and hence the faster it is moving, the slower it decays. But this is exactly what is expected from time dilation. We usually quote the rate in the rest frame of the decaying particle. For decays, the number left as a function of time is

$$N = N_0 e^{-t/\tau}$$

The rate per particle in the rest frame is then

$$\text{Rate} = -\frac{1}{N}\frac{dN}{dt} = \frac{1}{\tau} = \Gamma$$

Hence Fermi's Golden Rule directly gives us the width, or partial width if there are several channels, i.e.

$$\Gamma = |M|^2 \rho \frac{1}{2m_X}$$

For reactions, the rate is given by the cross section times the flux of particles

$$\sigma = \frac{\text{Rate}}{\text{Flux}}$$

Consider the reaction of a and b so

$$\sigma = \frac{1}{\text{Flux}}\left(|M|^2 \rho \frac{1}{4E_a E_b}\right)$$

The flux per particle for just two incoming particles is simply their relative velocity

$$\text{Flux} = |\boldsymbol{\beta}_a - \boldsymbol{\beta}_b|$$

where $\boldsymbol{\beta}_a$ and $\boldsymbol{\beta}_b$ are the velocity three-vectors of the incoming particles a and b. In the centre of mass system $\boldsymbol{p}_b = -\boldsymbol{p}_a$ so the flux becomes

$$\text{Flux} = \left|\frac{\boldsymbol{p}_a}{E_a} - \frac{\boldsymbol{p}_b}{E_b}\right| = |\boldsymbol{p}_a|\left(\frac{1}{E_a} + \frac{1}{E_b}\right) = p_a \frac{E_a + E_b}{E_a E_b} = \frac{p_a E_{cm}}{E_a E_b}$$

Hence

$$\sigma = |M|^2 \rho \frac{1}{4 p_a E_{cm}}$$

We will often work in the high energy limit, meaning $E \sim p \gg m$ and $\beta \sim 1$, so clearly in this limit

$$E_a = p_a = \frac{1}{2}\sqrt{s}$$

the flux is

$$\text{Flux} = 2$$

and the cross section becomes

$$\sigma = \frac{|M|^2 \rho}{2s}$$

9.3 Phase space

We still need to evaluate the phase space and the matrix element. The former gives the number of states available to the final particles produced in the decay or reaction; the more states there are to go to, the faster the reaction proceeds. You will have met the concept of density of states in first year structure of matter, where the number of states per unit element of momentum space was

$$dN = \frac{d^3p}{h^3} = \frac{d^3p}{(2\pi\hbar)^3} \to \frac{d^3p}{(2\pi)^3}$$

The general form of the relativistic phase space looks complicated

$$d\rho = (2\pi)^4 \int \delta^4(P^\mu_{in} - P^\mu_{out}) \prod_{out} \frac{d^3 p_{out}}{(2\pi)^3} \frac{1}{2E_{out}}$$

where the $\delta^4()$ here is the product of four Dirac delta functions, one for each of the total four-momentum components. There is a $d^3p/(2\pi^3)$ for every outgoing particle, so the number of integrals depends on the number of final particles. Again, the $1/2E$ is due to the normalisation, here of the outgoing particle wavefunctions.

This reduces to something manageable for a two body final state so consider two particles c and d. Again, in the high energy limit and working in the centre of mass so $\boldsymbol{p}_{in} = 0$, then the $\delta^3(\boldsymbol{p}_{in} - \boldsymbol{p}_{out})$ means $\boldsymbol{p}_{out} = \boldsymbol{p}_c + \boldsymbol{p}_d = 0$, so

$$\boldsymbol{p}_d = -\boldsymbol{p}_c, \qquad E_d = E_c$$

We can use one of the d^3p_{out} integrals, e.g. p_d, to get rid of these delta functions which leaves us with

$$d\rho_2 = \frac{1}{4(2\pi)^2}\int \delta(E_{cm} - 2E_c)\frac{d^3p_c}{E_c^2}$$

But in this limit

$$d^3p_c = p_c^2 dp_c d\Omega = E_c^2 dE_c d\Omega$$

where the element of solid angle

$$d\Omega = d(\cos\theta)d\phi$$

Hence

$$d\rho_2 = \frac{1}{16\pi^2}\int \delta(E_{cm} - 2E_c)dE_c d\Omega = \frac{1}{32\pi^2}d\Omega$$

or

$$\frac{d\rho_2}{d\Omega} = \frac{1}{32\pi^2}$$

in the high energy limit. Note, if masses are included, this becomes

$$\frac{d\rho_2}{d\Omega} = \frac{1}{32\pi^2}\sqrt{1 - \frac{2(m_c^2 + m_d^2)}{s} + \frac{(m_c^2 - m_d^2)^2}{s^2}}$$

Using the above in the high energy limit, for a two body decay $X \to c + d$, then

$$\frac{d\Gamma}{d\Omega} = \frac{|M|^2}{64\pi^2 m_X}$$

and for a cross section $a + b \to c + d$, then

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 s}$$

The three body or higher final states are not easy to do, even in the high energy limit; luckily we rarely need them on this course.

9.4 Matrix element

The matrix element contains the "physics" of the interaction and so must be calculated from the Lagrangian we obtained in the previous lecture. However, the general process of getting from a Lagrangian to a matrix element is more than a lecture course in its own right. Luckily Feynman came along and helped us; the system of Feynman diagrams and rules means we can apply physically intuitive rules to a Lagrangian to allow us to calculate the matrix element without needing to grind through the full rigours of quantum field theory. A Feynman diagram actually represents a term in a perturbation series and each item in it represents a multiplicative factor in the matrix element; by multiplying them all together, you can calculate the amplitude. There are three types of thing which appear in the diagrams

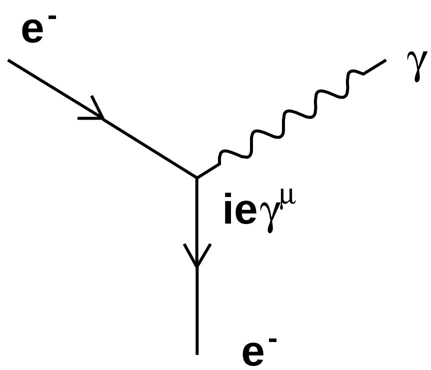
- The external lines
- The vertices
- The internal lines

Consider the first; the incoming and outgoing particles are known as the "external lines" because they go to the outside of the diagram. Since this is a perturbation calculation, these are the free, non-interacting states and so mathematically we use the Dirac or photon free particle solutions we obtained earlier.

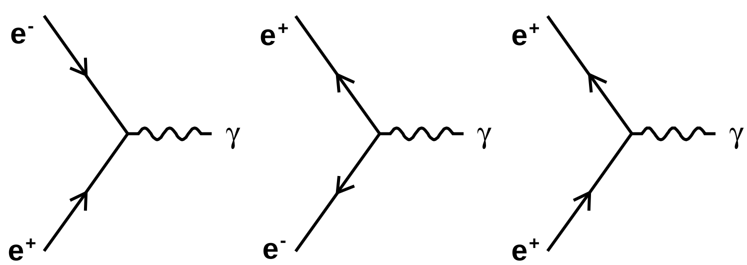
The vertices are where particles come together and actually interact. These clearly depend on the Lagrangian interaction term, which for QED we found was

$$-eA_\mu\overline{\psi}\gamma^\mu\psi$$

The way to interpret this is that there must be an incoming electron (ψ), an outgoing electron ($\overline{\psi}$) and a photon (A_μ) at every vertex and they are put together with an $ie\gamma^\mu$.



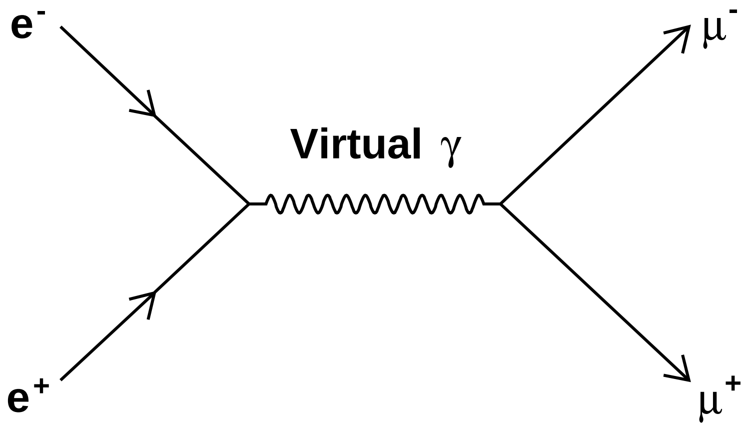
That is the only interaction, and hence vertex, allowed by this Lagrangian density. Note, the electrons could have negative energy and so the same term physically represents an incoming or outgoing positron (or both)



so this takes into account all the possible types of interaction. The photon is its own antiparticle, so the photon can be going in or out.

The above diagram would be the interaction with the highest rate in QED except for one small problem; it does not conserve energy and momentum. Think of this in one of the electron rest frames. Hence, in a real process, we must have at least two of the vertices in any diagram. As the number of vertices goes up, so does the power of e in the matrix element. Since e is smaller than one, the size of these amplitudes is reduced. Hence, the higher the number of vertices and more complicated the diagram, the smaller its contribution.

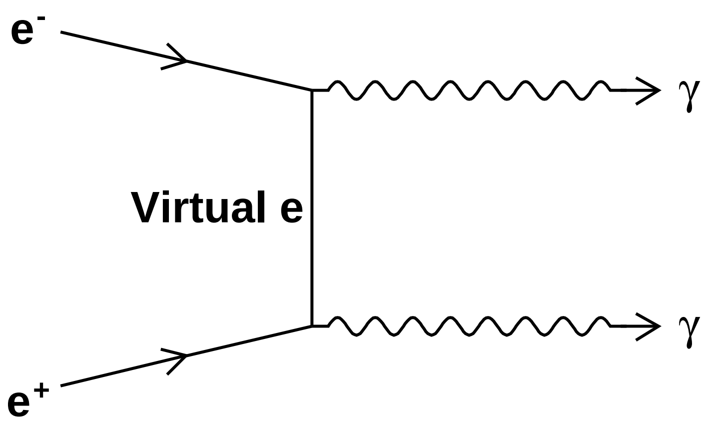
It should be clear that there will always be at least one particle, like the photon in the diagram, which is *not* external and only exists within the diagram. Such lines are therefore not solutions of the free particle equation and are called "virtual" or "off mass-shell" particles; their mathematical contribution to the matrix element is called the propagator. They have apparently



weird properties; for example, the photon does not have zero mass, meaning $E^2 - p^2 \neq 0$. We know its energy and momentum because those quantities are conserved at every vertex. This non-zero mass is allowed because only free photon solutions are required to have $p_\mu p^\mu = 0$; others have a different relation and in general, writing $p_\mu p^\mu = q^2$, then the photon propagator needed is

$$\frac{ig^{\mu\nu}}{q^2}$$

Electrons can also be virtual



and their propagator is

$$\frac{i\gamma^\mu p_\mu + m}{q^2 - m^2}$$

All these points are summarised in the Feynman rules in the handout.

It is worth trying to understand what is going on physically with the virtual particles. The internal lines correspond to the field being excited by the particles at either end of the line and basically are the factors which appear for these excited solutions. There is a direct and very close analogy to a simple harmonic oscillator. The equation can be written

$$\partial_0 \partial^0 x + \omega_0^2 x = F$$

for which the resemblance to the free Klein-Gordon equation should *not* be overlooked

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

We know if we excite the oscillator at a non-resonant frequency

$$F = F_0 e^{-i\omega t}, \qquad \omega \neq \omega_0$$

then we get a solution for the amplitude of

$$x_0 = \frac{F_0}{\sqrt{(\omega^2 - \omega_0^2)^2}}$$

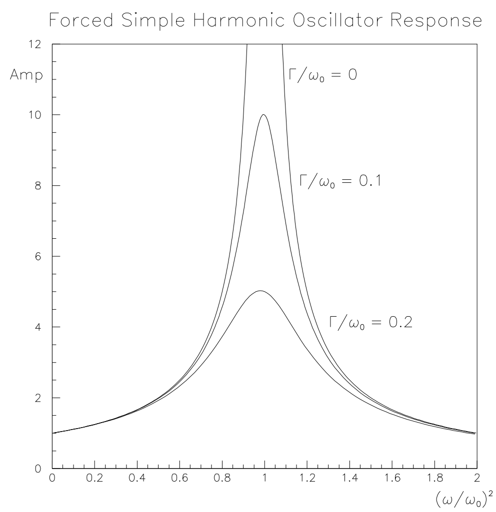
With a damping term, this becomes

$$\partial_0 \partial^0 x + \Gamma \partial^0 x + \omega_0^2 x = F$$

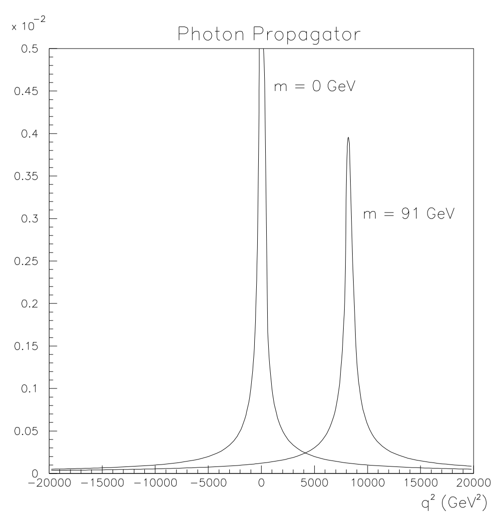
and the solution for the amplitude is

$$x_0 = \frac{F_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + \Gamma^2 \omega^2}}$$

This is a resonance shape.



The photon propagator is exactly analogous, where we now consider the mass rather than the frequency, so the terms become $\omega^2 - \omega_0^2 \to q^2 - m^2$ which is simply q^2 for the massless photon.



We can calculate the momentum dependence of the propagators quite simply. For the photon

$$\partial_\mu \partial^\mu A^\nu = [J^\nu]$$

where $[J^\nu]$ plays the role of the force F is the SHO case and is caused by the electrons at either end of the photon. Hence,

$$p_\mu p^\mu A^\nu = q^2 A^\nu = [J^\nu]$$

or

$$A^\nu \propto \frac{1}{q^2}$$

For the electron propagator,

$$i\gamma^\mu \partial_\mu \psi - m\psi = [eA_\mu \gamma^\mu \psi]$$

so

$$(i\gamma^\mu p_\mu - m)\psi = [eA_\mu \gamma^\mu \psi]$$

Here, we have matrices so we cannot divide directly but we can multiply by $i\gamma^\nu p_\nu + m$ from the left to give

$$(i\gamma^\nu p_\nu + m)(i\gamma^\mu p_\mu - m)\psi = (i\gamma^\mu p_\mu + m)[eA_\mu \gamma^\mu \psi]$$

but using the fundamental relation for the γ matrices, this product gives

$$(i\gamma^\nu p_\nu + m)(i\gamma^\mu p_\mu - m) = p^\mu p_\mu - m^2 = q^2 - m^2$$

so

$$\psi \propto \frac{i\gamma^\mu p_\mu + m}{q^2 - m^2}$$