APP - Lecture 8 - The QED Lagrangian

8.1 Introduction

We have used the Lagrangian formalism to find the free particle Lagrangian densities for the Dirac and Maxwell equations. We want to apply the formalism to the equations including the interaction terms, i.e.

$$i\gamma^\mu \partial_\mu \psi - m\psi - qA_\mu \gamma^\mu \psi = 0$$

and

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = J^\nu$$

and show they are consistent. We also then want to use the formalism to find conserved quantities.

8.2 Dirac Lagrangian density

The free Dirac Lagrangian density we used was

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$$

which resulted in the free Dirac equation.

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

as required. Can we add the EM potential? The extra term needed is $-q\gamma^\mu A_\mu \psi$ which cannot come from $\partial_\mu \left[\partial\mathcal{L}/\partial\left(\partial_\mu \overline{\psi}\right)\right]$ at it would then contain a derivative. Hence, we need to add a new term

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi - qA_\mu \overline{\psi}\gamma^\mu \psi$$

This is known as a Lagrangian "interaction term". Note, the potential A^μ here is not varying dynamically in response to the electron motion. It is a fixed potential imposed externally.

8.3 EM Lagrangian density

Similarly, we saw the Lagrangian density for the EM potentials for free photons was

$$\mathcal{L} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

This gives

$$-\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0$$

We now want to get the equation with current

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = J^\nu$$

Again there is no derivative, so we add an interaction to give

$$\mathcal{L} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - A_\mu J^\mu$$

so that the Euler-Lagrange equations become

$$-\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = -J^\nu$$

Again, J^ν is being imposed externally and it has no dynamics within the Lagrangian density.

Note, this Lagrangian density is (apparently) not gauge invariant any more.

8.4 Combined Lagrangian density

Can we put these together? It is clear we cannot have two different interaction terms, as that would give extra terms (for the other) in each equation. The interaction terms *are* compatible if we identify

$$J^\mu = q\overline{\psi}\gamma^\mu\psi$$

and the total Lagrangian density for the combined fields becomes

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - \frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - qA_\mu\overline{\psi}\gamma^\mu\psi$$

This is *not* a trivial result; this shows the two equations we got including interaction terms are compatible with each other.

Note, the dynamics of both fields are included in the same Lagrangian density so that each can change under the influence of the other. This results in a non-linear system which is usually only solved using perturbative techniques. These are the Feynman diagrams which we will discuss throughout the course.

8.5 Conserved quantities

When using fields, Nöther's theorem is also modified. Rather than having a conserved quantity if the Lagrangian is invariant under changes of a parameter α

$$\frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial \alpha}$$

the theorem instead now gives a conserved current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu q\right)}\frac{\partial q}{\partial \alpha}$$

which, for more than one field variable, is a sum over the q_j.

Does the above combined Lagrangian density have any invariants and hence symmetries which we haven't found yet? Besides the standard invariance to space and time which we discussed previously, it has an additional symmetry due to the Dirac field being complex. You all know that a constant phase is unobservable in quantum mechanics, so we need to look at what happens if we replace

$$\psi \to \psi e^{-i\alpha}, \qquad \overline{\psi} \to \overline{\psi}e^{i\alpha}$$

Clearly, the Lagrangian density is unchanged under this transformation. What is the Nöther current? We need

$$\frac{\partial\psi}{\partial\alpha} = -i\psi, \qquad \frac{\partial\overline{\psi}}{\partial\alpha} = i\overline{\psi}$$

Remembering there are two fields, ψ and $\overline{\psi}$, then the Nöther current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)} \frac{\partial \psi}{\partial \alpha} + \frac{\partial \overline{\psi}}{\partial \alpha} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \overline{\psi}\right)} = (i\overline{\psi}\gamma^\mu)(-i\psi) + (i\overline{\psi})(0) = \overline{\psi}\gamma^\mu\psi$$

i.e. the probability current which we already know is conserved. Note, this result can be obtained from either the free or the interaction Lagrangian density. This is called a "global phase transformation", as the phase change is the same everywhere. This rather odd result says that QM's insensitivity to changes in phase, i.e. the symmetry of phase changes, results in normalisation (and hence charge) conservation.

Why do we specify "global" here? What if α was not constant, but $\alpha = \alpha(x^\mu)$, i.e. it varied "locally"? Then the Lagrangian is no longer unchanged, as

$$\partial_\mu \psi \rightarrow \partial_\mu \psi - i(\partial_\mu \alpha)\psi$$

so the Lagrangian density becomes

$$\mathcal{L} \rightarrow \mathcal{L} + (\partial_\mu \alpha)\overline{\psi}\gamma^\mu\psi$$

Hence, this seems uninteresting because it is not a symmetry. However, what if we do a gauge transformation? Changing

$$A^\mu \rightarrow A^\mu + \partial^\mu \Lambda$$

gives

$$\mathcal{L} \rightarrow \mathcal{L} - q(\partial_\mu \Lambda)\overline{\psi}\gamma^\mu\psi$$

so if we choose $\alpha = q\Lambda$ and do both a local phase change *and* a gauge transformation at the same time, the Lagrangian is indeed unchanged. (Mundanely, It turns out the conserved current is unchanged.) So, this Lagrangian is invariant under a combined local phase/gauge transformation (often called just a local gauge transformation, with the phase implicit).

8.6 The gauge principle

This might seem vaguely interesting, but does it have any particular significance? If QED was the only force, then probably not, but it seems the other two forces we will consider also share the underlying idea. Let's invert the argument; given the combined free Lagrangian density for the Dirac and Maxwell equations, then we could ask what Lagrangian interaction term would keep the overall Lagrangian invariant under local gauge transformations. It turns out there is only one possible term, i.e.

$$-qA_\mu \overline{\psi}\gamma^\mu\psi$$

This interaction term is actually most easily found by doing the substitution we found before

$$\partial_\mu \psi \rightarrow D_\mu \psi = (\partial_\mu + iqA_\mu)\psi$$

into the Lagrangian density; the D_μ operator is locally gauge invariant itself, so this will always conserve gauge invariance.

If we impose gauge invariance as a principle, we determine the form of the interaction, and hence all the physics of QED. So, for the other forces, we can deduce the interaction terms using this principle once we know the particular symmetry of the force fields.

A few final points.

1. A mass term for the photon would be something like $m^2 A^\mu A_\mu/2$ but this cannot be made gauge invariant so the gauge principle requires the photon to be massless.

2. The full Lagrangian has an invariance under both $\hat{P}$ and $\hat{C}$ operations, meaning QED conserves both parity and charge conjugation quantum numbers.

3. Finally, when considering both electrons and muons, for example, we might postulate interaction terms containing things like $\overline{\psi}_e \gamma^\mu \psi_\mu$. These can also not be made gauge invariant as kinetic terms containing the same wavefunction, e.g. $i\overline{\psi}_e \gamma^\mu \partial_\mu \psi_e$, under local phase changes will only generate terms with the same wavefunction. To get a cross term would require a kinetic term like $i\overline{\psi}_e \gamma^\mu \partial_\mu \psi_\mu$ which would imply free μ and e spontaneously can change into each other. As we will see, the absence of interaction cross terms means the decay $\mu \rightarrow e + \gamma$ is not allowed and indeed has never been seen experimentally.