APP - Lecture 7 - Lagrangians

7.1 Introduction

You have all done classical mechanics in terms of Newton's Laws. There is an alternative approach which emphasises potentials rather than forces; this is the Lagrangian (and subsequently Hamiltonian) formalism. There is no new physics in the different method, but it is very useful for situations which involve symmetries and constrained systems, both of which we will be interested in.

7.2 The general formalism

We want to find the path of a particle $q(t)$ (where q is, for example, the x position) from an initial position $q_i(t_i)$ to a final position $q_f(t_f)$. We construct a function called a Lagrangian, L, which is (here assumed to be) a function of q and $dq/dt = \dot{q}$ only. To be concrete, for classical particles

$$L = T - V$$

(Note is is *not* $T + V$, which is the total energy.) We then construct a quantity called the action

$$A = \int_{t_i}^{t_f} L(q, \dot{q})\ dt$$

We want to find the "real" path taken by the particle. If we guess a function for $q(t)$, then we can calculate $\dot{q}$ and so do the integral (at least in principle). This will give us a value for the action A. We could then guess a different function and recalculate A, which will in general have a different value. A is said to be a "functional" of q, $A(q)$, i.e. a function of a function. Note, it is *not* a function of t as this is integrated out.

How do we know which is the real path? Of all the possible paths, the real one is the one which gives the smallest value of A. This is called the "Principle of Least Action". This principle is quite general and can be applied in many situations, not just those with a time dependence; e.g. the shape of a hanging rope can be calculated in a similar way. How do we find the actual function which gives the smallest A? If A was a simple function of a variable x, then we would clearly want $dA/dx = 0$. There is a similar calculus for functionals too and mechanically it works in a similar way. However, this "calculus of variations" is not done as part of the course, so we will work through this particular problem by hand. For a given path $q(t)$, let's shift it slightly by $\delta q(t)$, where we need to be sure $\delta q(t_i) = \delta q(t_f) = 0$. The change in A is given by

$$\delta A = \int_{t_i}^{t_f} \delta L\ dt = \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right]\ dt$$

Considering

$$\int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\ dt = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_i}^{t_f} = 0 = \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\right]\ dt$$

then we get

$$\delta A = \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] \delta q \; dt$$

But we need $\delta A = 0$ at the minimum and the only way this can be true for *any* $\delta q(t)$ is if

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

This is the Euler-Lagrange equation which allows us to calculate the path $q(t)$. For example, for classical one dimensional motion, with $q = x(t)$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$

so

$$\frac{\partial L}{\partial x} = -\frac{dV}{dx}, \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

so the Euler-Lagrange equation gives

$$-\frac{dV}{dx} = m\ddot{x}$$

which is the classical Newton force law. For more than one variable, $q_j(t)$, there is one Euler-Lagrange equation for each

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

with no cross terms. For example, for $q_j(t) = \boldsymbol{r}$, i.e. three dimensional motion, the Euler-Lagrange equations yield

$$-\boldsymbol{\nabla} V = m\ddot{\boldsymbol{r}}$$

as expected.

The above shows we can reproduce Newton's laws in this formalism; so what does using Lagrangians buy us?

- We can use non-inertial coordinates, unlike using Newton's laws. This can help simplify problems enormously.
- Lagrangians lead to quantisation quite straightforwardly.
- There is a systematic method for finding conserved quantities using Lagrangians.

The last point is extremely important; as an example, a very important conservation law follows from considering

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\dot{q}) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

This means

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = 0$$

The quantity in the brackets is the definition of the classical Hamiltonian, H, which is therefore shown to be a constant. For the one dimensional case above

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V = \frac{1}{2}m\dot{x}^2 + V$$

which is the total energy (which had better be conserved). The important point here is that the energy is *only* conserved because $L = L(q, \dot{q})$, not $L = L(q, \dot{q}, t)$, or there would have been an

Symmetry	Conserved Quantity
Translation in space (3 directions)	Momentum $\boldsymbol{p}$
Translation in time	Energy E
Rotation in space (3 angles)	Angular Momentum

Table 7.1: Space-time symmetries and their associated conserved quantities

extra $\partial L/\partial t$ term in the above. To rephrase this, L has an invariance to changes in time and this leads to energy conservation. Similarly, if L is invariant to changes in space, i.e. $V(x)$ is constant and does not depend on x, then $\partial L/\partial x = 0$ and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = 0$$

which is momentum conservation, as expected for a constant potential. The set of conserved quantities arising from space-time symmetries is shown in the table.
This method is very general, since for any parameter α

$$\frac{dL}{d\alpha} = \frac{\partial L}{\partial q}\frac{dq}{d\alpha} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{d\alpha} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\frac{dq}{d\alpha} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\left(\frac{dq}{d\alpha}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\frac{dq}{d\alpha}\right)$$

Hence, if L is invariant to changes in α, i.e. $dL/d\alpha = 0$, then the quantity

$$\frac{\partial L}{\partial \dot{q}}\frac{dq}{d\alpha}$$

is conserved. This is an example of a very general concept known as Nöther's theorem.

7.3 Lagrangians of fields

We can generalise this further; what about fields? Now the variable $q = q(t, \boldsymbol{r})$ so we need to make the Lagrangian by integrating over space

$$L = \int \mathcal{L}\left(q, \frac{\partial q}{\partial t}, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}, \frac{\partial q}{\partial z}\right)\, d^3r$$

where $\mathcal{L}$ is called the Lagrangian density (or often, sloppily, just called the Lagrangian again). The action is then

$$A = \int L\ dt = \int \mathcal{L}(q, \partial_\mu q)\ d^3r\ dt = \int \mathcal{L}(q, \partial_\mu q)\ d^4x$$

This action gives an Euler-Lagrange equation with extra terms, as there is more than just $\dot{q}$ now; the result of minimising A is that the Euler-Lagrange equation changes

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \qquad \rightarrow \qquad \frac{\partial \mathcal{L}}{\partial q} = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu q\right)}\right]$$

Let's take an example; consider the following with $q = \phi$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

(Note, this is apparently nothing like $T - V$ any more; this form has been chosen because it works.) Taking care of the fact that the derivative terms are really squares, this gives

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \qquad \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} = \partial^\mu \phi$$

so the Euler-Lagrange equation is

$$-m^2\phi = \partial_\mu \partial^\mu \phi$$

or

$$\partial_\mu \partial^\mu \phi + m^2\phi = 0$$

which is the Klein-Gordon equation.

7.4 Dirac Lagrangian density

The obvious question is; what Lagrangian density gives us the free particle Dirac equation? There are several forms (as the Lagrangian density is not unique) but a common one is

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$$

There is an additional complication here in that ψ is complex. There are then two variables, the real and imaginary parts, and in principle we need to do the Euler-Lagrange equations for each. However, there is a very useful mathematical trick; we can pretend the two variables are ψ and ψ^*. Clearly, this cannot really be true as

$$\left(\frac{\partial \mathcal{L}}{\partial \psi}\right)_{\psi^*}$$

is impossible; we cannot change ψ without changing ψ^*. But formally, we can proceed and it gives the right answer. In fact, we can even go further and use ψ and $\overline{\psi}$ as the two variables if we wish. This turns out to be the easiest thing to do, so taking derivatives with respect to $\overline{\psi}$, this gives

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi, \qquad \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \overline{\psi}\right)} = 0$$

so the Euler-Lagrange equation is clearly

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

as required. Note, we differentiate with respect to $\overline{\psi}$ to get the equation for ψ (and vice versa).

7.5 EM Lagrangian density

Similarly, we need a Lagrangian density for the EM potentials for free photons. The result is

$$\mathcal{L} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\left(E^2 - B^2\right)$$

This gives

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \qquad \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = -\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

so yielding for the Euler-Lagrange equations

$$-\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0$$

as required. This is the gauge independent form of the Maxwell equations. Note, since each of the $\partial^\mu A^\nu - \partial^\nu A^\mu$ terms itself is gauge invariant, the free Lagrangian is also. This is obvious as we can write it in terms of only the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields anyway.