Particle Physics Homework Assignment 4

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Problem 1: As shown in class the Dirac matrices must satisfy the anti-commutator relationships:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0 \quad \text{with} \quad \beta^2 = 1$$

I. Show that the α_i, β are Hermitian, traceless matrices with eigenvalues ± 1 and even dimensionality.

II. Show that, as long as the mass term is not zero and the matrix β is needed, there is no 2x2 set of matrices that satisfy all the above relationships. Hence, the Dirac matrices must be of dimension 4 or higher. First show that the set of matrices $(\mathbf{1}; \vec{\sigma})$ can be used to express any 2×2 matrix. That is the coefficients c_0, c_i always exist such that any 2×2 matrix can be written as:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = c_0 \mathbf{1} + c_i \sigma_i \quad (1)$$

Having shown this you can pick and intelligent choice for the α_i in terms of the Pauli matrices, for example $\alpha_i = \sigma_i$ which automatically obeys $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$, and express β in terms of $(\mathbf{1}; \vec{\sigma})$ using (1). Show then that there is no 2×2 β matrix that satisfies $\{\alpha_i, \beta\} = 0$.

Problem 2:

I. Show that the Weyl matrices:

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfy all the Dirac conditions of Problem 1. Hence, they form just another representation of the Dirac matrices, the Weyl representation, which is different than the standard Pauli-Dirac representation.

II. Show the the Dirac matrices in the Weyl representation are

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

III. Show that in the Weyl representation $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

IV. Solve the Dirac equation $[\vec{a}\cdot\vec{p}+\beta m]\Psi = E\Psi$ in the particle rest frame using the Weyl representation.

V. Compute the result of the chirality operators $\frac{(1\pm\gamma_5)}{2}$ when they are acting on the Dirac solutions in the Weyl representation.

Problem 3: Use the Dirac Hamiltonian in the standard Pauli-Dirac representation,

$$H = \vec{a}\cdot\vec{p}+\beta m$$

to compute $[H,\hat{L}]$ and $[H,\hat{\Sigma}]$ and show that they are not zero.
Use the results to show that:

$$[H,\hat{L}+(\frac{1}{2})\hat{\Sigma}] = 0$$

where the components of the angular momentum operator is given by:

$$\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$$

and the components of the spin operator are given by:

$$\hat{\Sigma}_i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

Recall that the Pauli matrices satisfy $\sigma^i\sigma^j = \delta^{ij}+i\epsilon^{ijk}\sigma^k$