Particle Physics Homework Assignment 9

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Problem 1: Write down the spinor for a positive energy neutrino of negative helicity and show explicitly that it is an eigenvector of both helicity and handedness (chirality).

Solution:

The mass of the neutrino is very small and can be assumed zero for the purposes of this exercise. Hence, the spinor for a negative helicity neutrino will be

$$\nu_L(x) = \sqrt{E}\begin{pmatrix}1\\ \vec{\sigma}\cdot\hat{p}\end{pmatrix}\chi^2 e^{-ip\cdot x} \quad \text{where} \quad \chi^2 = \begin{pmatrix}0\\ 1\end{pmatrix}$$

As in Homework Assignment 7 we can represent this schematically as shown in Figure 1.

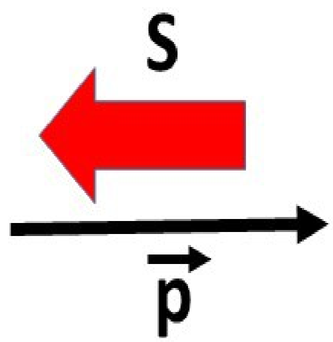

Figure 1: *A neutrino which moves with momentum* $\vec{p}$ *and has a polarization (helicity) which is opposite to the direction of motion.*

This spinor is an eigenvector of both helicity and handedness because:

Helicity:

$$\vec{\Sigma}\cdot\hat{p}\,\nu_L(x) = \begin{pmatrix}\vec{\sigma}\cdot\hat{p} & 0\\ 0 & \vec{\sigma}\cdot\hat{p}\end{pmatrix}\sqrt{E}\begin{pmatrix}1\\ \vec{\sigma}\cdot\hat{p}\end{pmatrix}\chi^2 e^{-ip\cdot x} = \sqrt{E}\begin{pmatrix}1\\ \vec{\sigma}\cdot\hat{p}\end{pmatrix}[\vec{\sigma}\cdot\hat{p}\chi^2]e^{-ip\cdot x}$$

One can choose for convenience the unit vector along the momentum direction, $\hat{p}$, to be along the z-axis and this way we have:

$$\vec{\sigma}\cdot\hat{p}\chi^2 = (-1)\chi^2$$

Hence,

$$\vec{\Sigma}\cdot\hat{p}\nu_L(x) = (-1)\nu_L(x)$$

and our chosen spinor has indeed negative helicity.

Handedness or Chirality:

$$\gamma_5\nu_L(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sqrt{E}\begin{pmatrix} 1 \\ \vec{\sigma}\cdot\hat{p} \end{pmatrix}\chi^2 e^{-ip\cdot x} = \sqrt{E}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} \\ 1 \end{pmatrix}\chi^2 e^{-ip\cdot x}$$

However,

$$(\vec{\sigma}\cdot\hat{p})(\vec{\sigma}\cdot\hat{p}) = \hat{p}\cdot\hat{p} = 1$$

Hence,

$$\gamma_5\nu_L(x) = \sqrt{E}\begin{pmatrix} 1 \\ \vec{\sigma}\cdot\hat{p} \end{pmatrix}\vec{\sigma}\cdot\hat{p}\chi^2 e^{-ip\cdot x} = (-1)\nu_L(x) \quad \text{(A)}$$

and the spinor has negative handedness also. This of course is no surprise. As we have shown in class, positive energy (particle) solutions of the Dirac equation with zero mass obey:

$$\gamma_5\Psi(x) = \Sigma\cdot\hat{p}\Psi(x)$$

In general we have that for any positive energy massless solution of the Dirac equation:

$$\Psi^{(+)}(x) = \sqrt{E}\begin{pmatrix} 1 \\ \vec{\sigma}\cdot\hat{p} \end{pmatrix}\chi^s e^{-ip\cdot x} \quad \text{with} \quad \chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{that:}$$

$$\gamma_5\Psi^{(+)}(x) = \pm\Psi^{(+)}(x) \qquad \Sigma\cdot\hat{p}\Psi^{(+)}(x) = \pm\Psi^{(+)}(x)$$

where the upper sign is for s = 1 and the lower sign is for s = 2.

For massless negative energy solutions as we have seen there is an extra minus. Consider a massless negative energy solution:

$$\Psi^{(-)}(x) = \sqrt{E}\begin{pmatrix} -\vec{\sigma}\cdot\hat{p} \\ 1 \end{pmatrix}\vec{\sigma}\cdot\hat{p}\chi^{s} e^{-ip\cdot x}$$

It is easy to show that here we have that:

$$\gamma_5\Psi^{(-)}(x) = \mp\Psi^{(-)}(x) \qquad\qquad \Sigma\cdot\hat{p}\,\Psi^{(-)}(x) = \pm\Psi^{(-)}(x)$$

where the upper sign is for s = 1 and the lower sign is for s = 2. This of course is a direct consequence to the fact that for negative energy solutions we have that:

$$\gamma_5\Psi(x) = -\Sigma\cdot\hat{p}\,\Psi(x)$$

Problem 2: If ν_L is a spinor describing neutrinos of negative helicity show that

$$\frac{1-\gamma_5}{2}\nu_L = \nu_L \text{ and } \frac{1+\gamma_5}{2}\nu_L = 0$$

Solution: Using the results from the previous problem we have:

$$\frac{1+\gamma_5}{2}\nu_L(x) = \frac{1+(-1)}{2}\nu_L(x) = 0$$

and

$$\frac{1-\gamma_5}{2}\nu_L(x) = \frac{1-(-1)}{2}\nu_L(x) = \nu_L(x)$$

Problem 3: Prove the identity:

$$\left(1 - \frac{\vec{\sigma}\vec{p}}{E+M}\right) = \frac{1}{2}\left(1 - \frac{p}{E+M}\right)(1 + \vec{\sigma}\hat{p}) + \frac{1}{2}\left(1+\frac{p}{E+M}\right)(1 - \vec{\sigma}\hat{p})$$

where $p = |\vec{p}|$ and $\hat{p} = \vec{p}/p$

Solution:

Let

$$\left(1 - \frac{\vec{\sigma}\vec{p}}{E+M}\right) = A\ \left(1 + \vec{\sigma}\hat{p}\right) + B\ \left(1 - \vec{\sigma}\hat{p}\right) \qquad (1)$$

We are entitled to do this because the Pauli matrices and the identity matrix form a basis. Using this and (1) we have:

$$1 = A + B \qquad (2)$$

$$-\frac{p}{E+M} = A - B \qquad (3)$$

From (2) and (3) we get that:

$$A = \frac{1}{2}\left(1 - \frac{p}{E+M}\right) \text{ and } B = \frac{1}{2}\left(1 + \frac{p}{E+M}\right) \qquad (4)$$

Finally (1) and (4) give:

$$\left(1 - \frac{\vec{\sigma}\vec{p}}{E+M}\right) = \frac{1}{2}\left(1 - \frac{p}{E+M}\right)\left(1 + \vec{\sigma}\hat{p}\right) + \frac{1}{2}\left(1+\frac{p}{E+M}\right)\left(1 - \vec{\sigma}\hat{p}\right)$$

Similarly one can also derive:

$$\left(1 + \frac{\vec{\sigma}\vec{p}}{E+M}\right) = \frac{1}{2}\left(1 + \frac{p}{E+M}\right)\left(1 + \vec{\sigma}\hat{p}\right) + \frac{1}{2}\left(1-\frac{p}{E+M}\right)\left(1 - \vec{\sigma}\hat{p}\right)$$

Problem 4: Show that the operator $\frac{(1 - \gamma_5)}{2}$ acting on a spinor describing a negative energy spin half massive fermion will result to a spinor with both positive helicity (right handed) and negative helicity (left handed) components. Further more show that the <u>left handed component is suppressed</u> by the ratio M/E. Hence, if the particle is massless it results to a pure right handed (positive helicity) spinor.

Solution:

We start by applying the projection operator on a negative energy solution:

$$\frac{(1 - \gamma_5)}{2}\Psi^{(-)} = \sqrt{|E|+M}\,\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ 1 \end{pmatrix}\chi^s e^{-ipx} \Rightarrow$$

$$\frac{(1 - \gamma_5)}{2}\Psi^{(-)} = \frac{1}{2}\sqrt{|E|+M}\begin{pmatrix} -1 \\ +1 \end{pmatrix}\left(1-\frac{\vec{\sigma}\cdot\vec{p}}{E-M}\right)\chi^s e^{-ipx} \qquad (1)$$

However, using previous results one can show that:

$$\left(1 - \frac{\vec{\sigma}\vec{p}}{E-M}\right) = \frac{1}{2}\left(1 - \frac{p}{E-M}\right)\left(1 + \vec{\sigma}\hat{p}\right) + \frac{1}{2}\left(1+\frac{p}{E-M}\right)\left(1 - \vec{\sigma}\hat{p}\right) \qquad (2)$$

Using (1) and (2) we have:

$$\frac{(1 - \gamma_5)}{2}\Psi^{(-)} = \frac{1}{4}\sqrt{|E|+M}\begin{pmatrix} -1 \\ +1 \end{pmatrix} \times$$

$$\times \left[\left(1 - \frac{p}{E-M}\right)\left(1 + \vec{\sigma}\hat{p}\right) + \left(1+\frac{p}{E-M}\right)\left(1 - \vec{\sigma}\hat{p}\right)\right]\chi^s e^{-ipx}$$

By taking in to account that the energy is a negative number we have that:

$$\frac{(1 - \gamma_5)}{2}\Psi^{(-)} = \frac{1}{4}\sqrt{|E|+M}\begin{pmatrix}-1\\+1\end{pmatrix} \times$$

$$\times \left[\left(1 + \frac{p}{|E|+M}\right)(1 + \vec{\sigma}\hat{p}) + \left(1-\frac{p}{|E|+M}\right)(1 - \vec{\sigma}\hat{p})\right]\chi^s e^{-ipx}$$

Hence, the spinor has both positive and negative helicity components. As seen here at high energy the positive helicity component dominates. During the lectures we have shown that the coefficient of the negative helicity component vanishes at high energy and approaches zero as $\sim M/E$.

Problem 5: Consider a massive chiral fermion given by:

$$\Psi_L = \frac{(1 - \gamma_5)}{2}\Psi$$

Define the polarization of the chiral massive fermion to be:

$$P = \frac{|\alpha_{RH}|^2-|\alpha_{LH}|^2}{|\alpha_{RH}|^2+|\alpha_{LH}|^2}$$

Where $|\alpha_{RH}|^2, |\alpha_{LH}|^2$ are the positive and negative helicity amplitudes of the fermion

Show that $P = -\frac{p}{E}=-\beta$ where ***p, E*** are the momentum and energy of the fermion.

Solution:

First we express the positive energy chiral fermion in terms of positive and negative helicity components as we did in the previous problem:

$$\frac{(1 - \gamma_5)}{2}\Psi^{(+)} = \sqrt{E+M}\,\frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M}\end{pmatrix}\chi^s e^{-ipx} \Rightarrow$$

$$\frac{(1 - \gamma_5)}{2}\Psi^{(+)} = \frac{1}{2}\sqrt{E+M}\begin{pmatrix}+1\\-1\end{pmatrix}\left(1-\frac{\vec{\sigma}\cdot\vec{p}}{E+M}\right)\chi^s e^{-ipx} \tag{1}$$

However,

$$\left(1 - \frac{\vec{\sigma}\,\vec{p}}{E-M}\right) = \frac{1}{2}\left(1 - \frac{p}{E-M}\right)\left(1 + \vec{\sigma}\,\hat{p}\right) + \frac{1}{2}\left(1+\frac{p}{E-M}\right)\left(1 - \vec{\sigma}\,\hat{p}\right) \tag{2}$$

Using (1) and (2) we get:

$$\frac{(1 - \gamma_5)}{2}\Psi^{(+)} = \frac{1}{4}\sqrt{E+M}\begin{pmatrix}+1\\-1\end{pmatrix} \times$$

$$\times \left[\left(1 - \frac{p}{E+M}\right)\left(1 + \vec{\sigma}\,\hat{p}\right) + \left(1+\frac{p}{E+M}\right)\left(1 - \vec{\sigma}\,\hat{p}\right)\right]\chi^s e^{-ipx}$$

Hence,

$$P = \frac{|\alpha_{RH}|^2-|\alpha_{LH}|^2}{|\alpha_{RH}|^2+|\alpha_{LH}|^2} = \frac{\left(1 - \frac{p}{E+M}\right)^2 - \left(1 + \frac{p}{E+M}\right)^2}{\left(1 - \frac{p}{E+M}\right)^2 + \left(1 + \frac{p}{E+M}\right)^2} \Rightarrow$$

$$P = \frac{-\frac{4\mathrm{p}}{E+M}}{2 + \frac{2\mathrm{p}^2}{(E+M)^2}} = \frac{-\frac{4\mathrm{p}}{E+M}}{2\left(1 + \frac{E^2-M^2}{(E+M)^2}\right)} = \frac{-\frac{4\mathrm{p}}{E+M}}{2\left(1 + \frac{E-M}{E+M}\right)} \Rightarrow$$

$$P = \frac{-\frac{4\mathrm{p}}{E+M}}{2\left(\frac{E+M + E-M}{E+M}\right)} = -\frac{p}{E}$$

As seen here the magnitude of the fermion polarization is simply equal to the relativistic boost of the particle in the laboratory frame which can be measured experimentally. This was used repeatedly by experiments in the past which measured the lepton polarization in weak decays in an effort to prove that the weak interaction is described by a V-A current-current Lagrangian.