Particle Physics, 4th year undergraduate, University of Ioannina

Particle Physics Homework Assignment 7

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Problem 1: Consider $\Psi = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$ to be solution of the Dirac equation where $u_A,\ u_B$ are two-component spinors. Show that in the non-relativistic limit where β is considerably smaller than 1, $u_B \sim \beta = v/c$.

Solution:

Consider the positive energy Dirac spinor:

$$u(E,\vec{p}) = \sqrt{E+M}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix}\chi^s$$

The lower element is depends upon velocity and this is the part that changes as velocity changes. This part needs to be re-written appropriately so we can study what happens at the non-relativistic limit:

$$\frac{\vec{\sigma}\cdot\vec{p}}{E+M} = \vec{\sigma}\cdot\hat{p}\times\frac{p}{M}\times\frac{1}{1+\sqrt{1+\frac{P^2}{M^2}}} \approx \vec{\sigma}\cdot\hat{p}\times\frac{p}{M}\times\frac{1}{1+1+\frac{P^2}{2M^2}} \Rightarrow$$

$$\frac{\vec{\sigma}\cdot\vec{p}}{E+M} \approx \vec{\sigma}\cdot\hat{p}\times\frac{p}{2M}\times\frac{1}{1+\frac{P^2}{4M^2}} \approx \vec{\sigma}\cdot\hat{p}\times\frac{p}{2M}\times(1-\frac{P^2}{4M^2})$$

At the non-relativistic limit the momentum is much smaller than the mass and the ratio of momentum over mass is a small number. So lets decide to compute this result of order $\frac{p}{M}$. Hence, we can drop higher order terms to get

$$\frac{\vec{\sigma}\cdot\vec{p}}{E+M} \approx \vec{\sigma}\cdot\hat{p}\times\frac{p}{2M} = \vec{\sigma}\cdot\hat{p}\frac{M\gamma\beta}{2M} \approx \vec{\sigma}\cdot\hat{p}\frac{\beta}{2}$$

Therefore, at the non-relativistic limit we have that:

$$u_{NR}(E,\vec{p}) = \sqrt{2M}\begin{pmatrix} 1 \\ \vec{\sigma}\cdot\hat{p}\,\frac{\beta}{2} \end{pmatrix}\chi^s$$

This is the reason that for particles (positive energy solutions) in the non-relativistic limit the lower component is smaller than then upper and it is referred to as ***the small component.***

Problem 2: Show that at the non-relativistic limit the motion of a spin half fermion of charge e at the presence of an electromagnetic field $A^\mu = (A^0;\vec{A})$ is described by:

$$\left[\frac{(\vec{p}-e\vec{A})^2}{2m} - \frac{e}{2m}\vec{\sigma}\cdot\vec{B} + eA^0\right]\chi = E\chi$$

where $\vec{B}$ is the magnetic field, σ^i are the Pauli matrices and $E=p^0-m$. Identify the g-factor of the fermion and show that the Dirac equation predicts the correct gyromagnetic ratio for the fermion. To write down the Dirac equation at the presence of an electromagnetic field substitute: $p^\mu \to p^\mu - eA^\mu$.

Solution:

The Dirac equation at the presence of an electromagnetic field can be written as:

$$\left[\gamma^\mu(P_\mu - eA_\mu) - m\right]u(\vec{p}) = 0 \qquad (1)$$

where P_μ, $\vec{p}$ are the energy momentum operator and the vector momentum. From (1) we have that:

$$\left[\gamma^0 P_0 - \vec{\gamma}\cdot\vec{p} - e\gamma^0 A_0 + e\vec{\gamma}\cdot\vec{A} - m\right]u(\vec{p}) = 0 \Rightarrow$$

$$\begin{bmatrix} p_0 - m - eA_0 & -\vec{\sigma}\cdot\vec{p} + e\vec{\sigma}\cdot\vec{A} \\ +\vec{\sigma}\cdot\vec{p} - e\vec{\sigma}\cdot\vec{A} & -p_0 - m + eA_0 \end{bmatrix}\begin{pmatrix}\chi \\ \varphi\end{pmatrix} = 0 \Rightarrow$$

$$(p_0 - m - eA_0)\chi - \vec{\sigma}\cdot(\vec{p} - e\cdot\vec{A})\varphi = 0 \qquad (2)$$

and

$$\vec{\sigma}\cdot(\vec{p} - e\cdot\vec{A})\chi - (p_0 + m - eA_0)\varphi = 0 \qquad (3)$$

In the non-relativistic approximation terms $\sim A_0 \varphi$ can be neglected. This uses the results form the previous problem (small component) and assumes that the field is weak. Hence, from (3) we have that:

$$\varphi = \frac{\vec{\sigma}\cdot(\vec{p}-e\cdot\vec{A})}{p_0+m}\chi \tag{4}$$

and by substituting into (2) we get:

$$(P_0-m-eA_0)\chi = \frac{\vec{\sigma}\cdot(\vec{p}-e\cdot\vec{A})\vec{\sigma}\cdot(\vec{p}-e\cdot\vec{A})}{p_0+m}\chi \tag{5}$$

$$\vec{\sigma}\cdot(\vec{p}-e\cdot\vec{A})\vec{\sigma}\cdot(\vec{p}-e\cdot\vec{A}) = \sigma_i(\vec{p}-e\cdot\vec{A})_i\sigma_j(\vec{p}-e\cdot\vec{A})_j \Rightarrow$$

$$\vec{\sigma}\cdot(\vec{p}-e\cdot\vec{A})\vec{\sigma}\cdot(\vec{p}-e\cdot\vec{A}) = \vec{p}^2-e\sigma_i\sigma_j(p_iA_j+A_ip_j)+e^2\vec{A}^2 \tag{6}$$

Note that the momenta and vector potentials do not commute since the momentum is an operator $p_i=-i\partial_i$.

$$\sigma_i\sigma_j(p_iA_j+A_ip_j)\Psi = (\delta_{ij}+i\varepsilon_{ijk}\sigma_k)(p_iA_j+A_ip_j)\Psi \Rightarrow$$

$$\sigma_i\sigma_j(p_iA_j+A_ip_j)\Psi = (\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p})\Psi+i\varepsilon_{ijk}\sigma_kp_iA_j\Psi+i\varepsilon_{ijk}\sigma_kA_ip_j\Psi \Rightarrow$$

$$\sigma_i\sigma_j(p_iA_j+A_ip_j)\Psi = (\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p})\Psi+i\varepsilon_{ijk}\sigma_k(-i\partial_i)A_j\Psi+i\varepsilon_{ijk}\sigma_kA_i(-i\partial_j)\Psi$$

by executing the differentiation of the product and canceling the relevant terms we get:

$$\sigma_i\sigma_j(p_iA_j+A_ip_j)\Psi = (\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p})\Psi+\sigma_k\varepsilon_{kij}(\partial_iA_j)\Psi \Rightarrow$$

$$\sigma_i\sigma_j(p_iA_j+A_ip_j)\Psi = [\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}+\vec{\sigma}\cdot\vec{B}]\Psi \tag{7}$$

where the last term results of a coupling between the spin and the magnetic field. From (5) (6) and (7) we have that

$$(P_0-m-eA_0)\chi = \frac{\vec{p}^2-e[\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}+\vec{\sigma}\cdot\vec{B}]+e^2\vec{A}^2}{p_0+m}\chi \Rightarrow$$

$$(P_0-m-eA_0)\chi = \frac{(\vec{p}-e\vec{A})^2-e\vec{\sigma}\cdot\vec{B}}{p_0+m}\chi \Rightarrow$$

$$\frac{(\vec{p}-e\vec{A})^2-e\vec{\sigma}\cdot\vec{B}}{p_0+m}\chi+eA_0\chi \;=\; (p_0-m)\chi$$

which at the non-relativistic limit can be written as:

$$\left[\frac{(\vec{p}-e\vec{A})^2}{2m}-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}+eA_0\right]\chi \;=\; (p_0-m)\chi \;=\; E_{NR}\chi$$

As seen here, and this is the most important point of this exercise, the spin term which was put 'by hand' in non-relativistic quantum mechanics, it emerges naturaly when one considera a relativistic invariant equation. **Hence, spin is a relativistic effect which is predicted by a relativistic description of the interaction of an electromagentic field with an electron.**

Problem 3: Show that:

(a) $\bar{\Psi}\gamma_5\Psi$ is a pseudoscalar.
(b) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi$ is an axial vector.

Comment on the Lorentz and parity properties of the quantities:

(a) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi\bar{\Psi}\gamma_{\mu}\Psi$
(b) $\bar{\Psi}\gamma_5\Psi\bar{\Psi}\gamma_5\Psi$
(c) $\bar{\Psi}\Psi\bar{\Psi}\gamma_5\Psi$
(d) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi\bar{\Psi}\gamma_5\gamma_{\mu}\Psi$
(e) $\bar{\Psi}\gamma^{\mu}\Psi\bar{\Psi}\gamma_{\mu}\Psi$

It is given that $\{\gamma_{5,}\gamma^{\mu}\} \;=\; 0$.

Solution:

(a) Under Lorentz we have that:

$$\bar{\Psi}'(x')\gamma_5\Psi'(x') \;=\; \Psi'^{+}(x')\gamma^0\gamma_5\Psi'(x') \;=\; \Psi^{+}(x)S^{+}(\Lambda)\gamma^0\gamma_5 S(\Lambda)\Psi(x) \;\Rightarrow$$

$$\bar{\Psi}'(x')\gamma_5\Psi'(x') \;=\; \Psi^{+}(x)\gamma^0\gamma^0 S^{+}(\Lambda)\gamma^0\gamma_5 S(\Lambda)\Psi(x)$$

But we know that: $S^{-1}(\Lambda) \;=\; \gamma^0 S^{+}(\Lambda)\gamma^0$.

Hence,

$$\bar{\Psi}'(x')\gamma_5\Psi'(x') = \bar{\Psi}(x)S^{-1}(\Lambda)\gamma_5 S(\Lambda)\Psi(x)$$

The Lorentz transformation for spinors, S, can be written in terms of powers of $\gamma^\mu\gamma^\nu$ pairs. The matrix γ_5 anti-commutes which each of the gamma matrices. Hence, γ_5 commutes with the product of $\gamma^\mu\gamma^\nu$ pairs. Therefore γ_5 commutes with S and we have that:

$$\bar{\Psi}'(x')\gamma_5\Psi'(x') = \bar{\Psi}(x)\gamma_5\Psi(x)$$

So it is Lorentz invariant.

Next check parity:

$$\bar{\Psi}'(x')\gamma_5\Psi'(x') = \Psi'^{+}(x')\gamma^0\gamma_5\Psi'(x') = \Psi^{+}(x)\gamma^0\gamma^0\gamma_5\gamma^0\Psi(x) \Rightarrow$$

$$\bar{\Psi}'(x')\gamma_5\Psi'(x') = -\bar{\Psi}(x)\gamma_5\gamma^0\gamma^0\Psi(x) = -\bar{\Psi}(x)\gamma_5\Psi(x)$$

Hence, it is odd under parity. Quantities which are Lorentz invariant but odd under parity are called pseudo-scalars.

Hence, $\bar{\Psi}(x)\gamma_5\Psi(x)$ **is a pseudo-scalar.**

(b) Next consider the Lorentz transformation of the axial current:

$$J_5^{\mu}{}' = \bar{\Psi}'(x')\gamma_5\gamma^\mu\Psi'(x') = [\Psi'(x')]^{+}\gamma^0\gamma_5\gamma^\mu\Psi'(x') \Rightarrow$$

$$J_5^{\mu}{}' = [\Psi(x)]^{+}S^{+}(\Lambda)\gamma^0\gamma_5\gamma^\mu S(\Lambda)\Psi(x) = \bar{\Psi}(x)\gamma^0 S^{+}(\Lambda)\gamma^0\gamma_5\gamma^\mu S(\Lambda)\Psi(x) \Rightarrow$$

$$J_5^{\mu}{}' = \bar{\Psi}(x)S^{-1}(\Lambda)\gamma_5 S(\Lambda)S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\Psi(x) = \bar{\Psi}(x)\gamma_5\Lambda^\mu_\nu\gamma^\nu\Psi(x) \Rightarrow$$

$J_5^{\mu}{}' = \Lambda^\mu_\nu \quad \bar{\Psi}(x)\gamma_5\gamma^\nu\Psi(x) \Rightarrow$ **Hence, it transforms as a vector under Lorentz.**

Under parity you we get:

$$J_5^{\mu}{}' = \bar{\Psi}'(x')\gamma_5\gamma^\mu\Psi'(x') = [\Psi(x)]^{+}\gamma^0\gamma^0\gamma_5\gamma^\mu\gamma^0\Psi(x) \Rightarrow$$

$$J_5^{\mu}{}' = \bar{\Psi}(x)\gamma^0\gamma_5\gamma^\mu\gamma^0\Psi(x) = (-1)\bar{\Psi}(x)\gamma_5\gamma^0\gamma^\mu\gamma^0\Psi(x) \Rightarrow$$

$$J_5^{\mu}{}' = (-1)\bar{\Psi}(x)\gamma_5(\gamma^{\nu})^{+}\Psi(x) \Rightarrow$$

$$J_5^{0}{}' = (-1)J_5^{0}$$

and

$$\vec{J}_5{}' = (+1)\vec{J}_5$$

So J_5^{μ} transforms under Lorentz as a vector but under parity it transforms in a way that it is opposite from that of a polar vector. **Hence, J_5^{μ} is a axial vector.**

Answering the last part should be easy:

(a) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi\bar{\Psi}\gamma_{\mu}\Psi$: is Lorentz invariant because is it the dot product of a polar vector and an axial vector. For the same reason it has negative parity.

(b) $\bar{\Psi}\gamma_5\Psi\bar{\Psi}\gamma_5\Psi$: Obviously Lorentz invariant and even under parity as a product of two pseudo-scalars each having odd parity.

(c) $\bar{\Psi}\Psi\bar{\Psi}\gamma_5\Psi$: Lorentz invariant with odd parity (scalar times pseudo-scalar).

(d) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi\bar{\Psi}\gamma_5\gamma_{\mu}\Psi$: Dot product of two axial vectors will be of course Lorentz invariant with even parity.

(e) $\bar{\Psi}\gamma^{\mu}\Psi\bar{\Psi}\gamma_{\mu}\Psi$: Dot product of two polar vectors is both Lorentz invariant and even under parity.

Problem 4: Let P be the parity operator acting on Dirac spinors such that:

$$P\Psi(x^{\mu}.p^{\mu}) = e^{i\varphi}\gamma^{0}\Psi(x^{\mu},p^{\mu})$$

Show that:

$$P\Psi^{(+)}(x^{\mu},p^{\mu}) = +\Psi^{(+)}(x^{\mu}{}',p^{\mu}{}')$$

and

$$P\Psi^{(-)}(x^{\mu},p^{\mu}) = -\Psi^{(-)}(x^{\mu}{}',p^{\mu}{}')$$

where $\Psi^{(+)}$, $\Psi^{(-)}$ are the positive and negative energy solutions of the Dirac equation and $x^{\mu} = (x^0;\vec{x})$, $p^{\mu} = (p^0;\vec{p})$, $x^{\mu}{}' = (x^0;-\vec{x}) = (x^0;\vec{x}\,')$,

$$p^{\mu}{}' = (p^0;-\vec{p}) = (p^0;\vec{p}\,') \; .$$

Solution:

<u>Positive Energy Solutions:</u>

$$\Psi^{(+)}(x^\mu, p^\mu) = \sqrt{E+M}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix}\chi^s e^{-i\,p\cdot x} \Rightarrow$$

$$P\,\Psi^{(+)}(x^\mu, p^\mu) = \Psi^{(+)}{}'(x^\mu{}', p^{\mu'}) = \sqrt{E+M}\,\gamma^0\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix}\chi^s e^{-i\,p^0 x^0 - i\,\vec{p}\cdot\vec{x}\,'} \Rightarrow$$

$$P\,\Psi^{(+)}(x^\mu, p^\mu) = \Psi^{(+)}{}'(x^\mu{}', p^{\mu'}) = \sqrt{E+M}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot(-\vec{p})}{E+M} \end{pmatrix}\chi^s e^{-i\,p^0 x^0 + i(-\vec{p})\cdot\vec{x}\,'} \Rightarrow$$

$$P\,\Psi^{(+)}(x^\mu, p^\mu) = \sqrt{E+M}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}\,'}{E+M} \end{pmatrix}\chi^s e^{-i\,p'\cdot x'} = \Psi^{(+)}(x^\mu{}', p^\mu{}')$$

<u>Negative Energy Solutions</u>

$$\Psi^{(-)}(x^\mu, p^\mu) = \sqrt{|E|+M}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ 1 \end{pmatrix}\chi^s e^{-i\,p\cdot x} \Rightarrow$$

$$P\,\Psi^{(-)}(x^\mu, p^\mu) = \Psi^{(-)}{}'(x^\mu{}', p^\mu{}') = \sqrt{|E|+M}\,\gamma^0\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ 1 \end{pmatrix}\chi^s e^{-i\,p^0 x^0 + i\,\vec{p}\cdot\vec{x}} =$$

$$\sqrt{|E|+M}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ -1 \end{pmatrix}\chi^s e^{-i\,p^0 x^0 - i\,\vec{p}\cdot\vec{x}\,'} = \sqrt{|E|+M}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ -1 \end{pmatrix}\chi^s e^{-i\,p^0 x^0 + i(-\vec{p})\cdot\vec{x}\,'} =$$

$$-\sqrt{|E|+M}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot(-\vec{p})}{E-M} \\ +1 \end{pmatrix}\chi^s e^{-i\,p'\cdot x'} = -\sqrt{|E|+M}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}\,'}{E-M} \\ +1 \end{pmatrix}\chi^s e^{-i\,p'\cdot x'} = -\Psi^{(-)}(x^\mu{}', p^\mu{}')$$

In other words, as expected, the parity transformed spinor is a spinor that moves opposite than the original as shown in Figure 1.

Figure 1: *The effect of the parity operation on a spinor with momentum* $\vec{B}$ *. As seen here parity reverses the momentum vector but does not affect the spin.*

Problem 5: Consider the Dirac Hamiltonian:

$$\hat{H} \;=\; -i\,\vec{\alpha}\cdot\vec{\nabla} \;+\; \beta m \quad .$$

Show that:

$$[\hat{H}, \gamma^0 \hat{\Pi}] \;=\; 0$$

where, $\hat{\Pi}$ is the coordinate parity operator such that $\hat{\Pi}\, f(\vec{r}) \;=\; f(-\vec{r})$.

Solution:

$$[\hat{H}, \gamma^0 \hat{\Pi}]\Psi(\vec{r}) \;=\; -i\hbar\vec{\alpha}\cdot\vec{\nabla}\gamma^0\hat{\Pi}\,\Psi(\vec{r}) - \gamma^0\hat{\Pi}(-i\hbar\vec{\alpha}\cdot\vec{\nabla})\Psi(\vec{r}) \;\Rightarrow$$

$$[\hat{H}, \gamma^0 \hat{\Pi}]\Psi(\vec{r}) \;=\; -i\hbar\vec{\alpha}\cdot\vec{\nabla}\gamma^0\Psi(-\vec{r}) - \gamma^0(+i\hbar\vec{\alpha}\cdot\vec{\nabla})\Psi(-\vec{r}) \;\Rightarrow$$

$$[\hat{H}, \gamma^0 \hat{\Pi}]\Psi(\vec{r}) \;=\; -i\hbar\vec{\alpha}\cdot\vec{\nabla}\gamma^0\Psi(-\vec{r}) - (-i\hbar\vec{\alpha}\cdot\vec{\nabla})\gamma^0\Psi(-\vec{r}) \;\Rightarrow$$

$$[\hat{H}, \gamma^0 \hat{\Pi}]\Psi(\vec{r}) \;=\; 0$$

Problem 6: Consider the Dirac equation for an electron which couples to the Electromagnetic field

$$\left[i\gamma^{\mu}(\partial_{\mu}-ieA_{\mu}(x))-m\right]\Psi(x) = 0$$

where $A^{\mu}(x) = (\Phi(x);\vec{A}(x))$ is the electromagnetic field. Show that the Dirac equation is invariant under parity provided that the electron and the electromagnetic field transform under parity as follows.

$$P\,\Psi(\vec{x},t) = \Psi'(\vec{x}',t) = e^{i\varphi}\gamma^{0}\Psi(-\vec{x}',t)$$

$$P\,\Phi(\vec{x},t) = \Phi'(\vec{x}',t) = \Phi(-\vec{x}',t) = \Phi(\vec{x},t)$$

$$P\,\vec{A}(\vec{x},t) = \vec{A}'(\vec{x}',t) = -\vec{A}(-\vec{x}',t)$$

Solution:

$$\left[i\gamma^{\mu}(\partial_{\mu}-ieA_{\mu}(x))-m\right]\Psi(x) = 0 \Rightarrow$$

$$\left[i\gamma^{0}\partial_{0}+i\vec{\gamma}\cdot\vec{\nabla}+e\gamma^{0}\Phi(\vec{x},t)-e\vec{\gamma}\cdot\vec{A}(\vec{x},t)-m\right]\Psi(\vec{x},t) = 0 \Rightarrow \qquad (1)$$

$$\left[i\gamma^{0}\partial_{0}-i\vec{\gamma}\cdot\vec{\nabla}'+e\gamma^{0}\Phi(-\vec{x}',t)-e\vec{\gamma}\cdot\vec{A}(-\vec{x}',t)-m\right]\Psi(-\vec{x}',t) = 0 \Rightarrow$$

$$\left[i\gamma^{0}\partial_{0}+i\vec{\gamma}\cdot\vec{\nabla}'+e\gamma^{0}\Phi(-\vec{x}',t)+e\vec{\gamma}\cdot\vec{A}(-\vec{x}',t)-m\right]\gamma^{0}\Psi(-\vec{x}',t) = 0 \qquad (2)$$

By comparing (1) with (2) we see that Dirac's equation with electromagnetic coupling remains invariant if

$$\Psi'(\vec{x}',t) = e^{i\varphi}\gamma^{0}\Psi(-\vec{x}',t) \qquad (3)$$

$$\Phi'(\vec{x}',t) = \Phi(-\vec{x}',t) = \Phi(\vec{x},t) \qquad (4)$$

$$\vec{A}'(\vec{x}',t) = -\vec{A}(-\vec{x}',t) \qquad (5)$$

As seen here the spinor transforms as expected and the electromagnetic field transforms as a polar vector.