Particle Physics Homework Assignment 6

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Problem 1: Show that: $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}+i\vec{\sigma}(\vec{a}\times\vec{b})$

Solution:

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = a_i\sigma_i b_j\sigma_j = a_i b_j\sigma_i\sigma_j \tag{1}$$

$$\sigma_i\sigma_j = \delta_{ij}+i\varepsilon_{ijk}\sigma_k \tag{2}$$

$$\Rightarrow a_i b_j(\delta_{ij}+i\varepsilon_{ijk}\sigma_k) = a_i b_i + i\sigma_k\varepsilon_{kij}a_i b_j \Rightarrow$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}+i\vec{\sigma}\cdot\vec{a}\times\vec{b}$$

Problem 2:

1. Solve the Dirac equation $[\vec{a}\cdot\vec{p}+\beta m]\Psi = E\Psi$ in the particle rest frame using the Weyl representation.
2. Compute the result of the chirality operators $\dfrac{(1\pm\gamma_5)}{2}$ when they are acting on the solutions of the Dirac equation expressed in the Weyl representation.

Solution:

1. At the particle rest frame the momentum is zero so we have that:

$$[\vec{a}\cdot\vec{p}+\beta m]\Psi = E\Psi \Rightarrow \beta\Psi = E\Psi \Rightarrow \begin{pmatrix}-E & m \\ m & -E\end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = 0 \tag{1}$$

We require that (1) has non-trivial (non-zero) solutions. Therefore the determinant must be zero. This way the inverse matrix does not exist because if it did one could always multiply with the inverse from the left side and prove that the solution is identically zero.

This means that: $E^2 = m^2 \Rightarrow E = \pm m$

and again we have negative and positive solutions.

<u>Positive Energy Solutions:</u>

To obtain the positive energy spinors we substitute $\boldsymbol{E = +m}$ in equation (1) :

$$\begin{pmatrix} -m & m \\ m & -m \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \Rightarrow \quad \chi_2 = \chi_1$$

Hence:

$$\Psi = \begin{pmatrix} \chi \\ \chi \end{pmatrix} \Rightarrow \Psi^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; \Psi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

<u>Negative Energy Solutions:</u>

In a similar fashion the negative energy solutions can be obtained by substituting $\boldsymbol{E = -m}$ in equation (1) :

$$\begin{pmatrix} m & m \\ m & m \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 = 0 \Rightarrow \quad \chi_2 = -\chi_1$$

and

$$\Psi = \begin{pmatrix} +\chi \\ -\chi \end{pmatrix} \Rightarrow \Psi^3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \; ; \; \Psi^4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

2. The chirality operators are given by:

$$\frac{(1+\gamma_5)}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{(1-\gamma_5)}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence,

$$\frac{(1+\gamma_5)}{2}\Psi^{1,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \chi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \qquad \frac{(1+\gamma_5)}{2}\Psi^{3,4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} +\chi \\ -\chi \end{pmatrix} = -\begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

$$\frac{(1-\gamma_5)}{2}\Psi^{1,2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \chi \end{pmatrix} = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \qquad \frac{(1-\gamma_5)}{2}\Psi^{3,4} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} +\chi \\ -\chi \end{pmatrix} = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$$

Notice that the chirality operators in the Weyl representation do not mix upper and lower components as they do in the Pauli-Dirac representation.

Problem 3: Positive energy solutions of the Dirac Equation correspond to the 4-vectror current: $J^{\mu} = 2p^{\mu} = 2(E\,;\vec{p});\quad E>0$. Show that negative energy solutions correspond to the current $J^{\mu} = -2(E\,;\vec{p}) = 2(|E|;-\vec{p}) = -2p^{\mu};\quad E<0$.

Solution:

We have already shown during the lectures that for positive energy solutions of the Dirac equation one can compute the current:

$$J^{\mu} = \bar{\Psi}\gamma^{\mu}\Psi = 2P^{\mu}$$

where P^{μ} is the 4-momentum of the fermion.

For negative energy solutions we have shown that the 0th component of the current $J^{\mu} = \bar{\Psi}\gamma^{\mu}\Psi$ is given by:

$$J^{0} = \bar{\Psi}\gamma^{0}\Psi = |N|^{2}\frac{2E}{E-M} = |N|^{2}\frac{-2|E|}{-|E|-M} \geq 0 \quad \text{for } E<0$$

Next lets compute the vector part of the 4-current:

$$\vec{J} = \bar{\Psi}\vec{\gamma}\Psi = |N|^{2}\ \Psi^{+}\gamma^{0}\vec{\gamma}\ \Psi \Rightarrow$$

$$\vec{J} = |N|^{2}(\chi^{s})^{+}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E-M}, & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E-M}\\ 1\end{pmatrix}\chi^{s} \Rightarrow$$

$$\vec{J} = |N|^{2}(\chi^{s})^{+}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E-M}, & 1\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E-M}\\ 1\end{pmatrix}\chi^{s} \Rightarrow$$

$$\vec{J} = |N|^{2}(\chi^{s})^{+}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E-M}, & 1\end{pmatrix}\begin{pmatrix}\vec{\sigma}\chi^{s}\\ \vec{\sigma}\frac{\vec{\sigma}\cdot\vec{p}}{E-M}\chi^{s}\end{pmatrix} \Rightarrow$$

$$\vec{J} = |N|^{2}(\chi^{s})^{+}\left[\frac{\vec{\sigma}\cdot\vec{p}}{E-M}\vec{\sigma} + \vec{\sigma}\frac{\vec{\sigma}\cdot\vec{p}}{E-M}\right]\chi^{s} \Rightarrow$$

$$J^{i} = |N|^{2}(\chi^{s})^{+}\left[\frac{\sigma^{l}p^{l}}{E-M}\sigma^{i} + \sigma^{i}\frac{\sigma^{l}p^{l}}{E-M}\right]\chi^{s} \Rightarrow$$

$$J^i = |N|^2 (\chi^s)^+ \left[\sigma^l \sigma^i + \sigma^i \sigma^l\right] \chi^s \frac{p^l}{E-M} \qquad \Rightarrow$$

$$J^i = |N|^2 (\chi^s)^+ \, 2\delta_{il} \chi^s \frac{p^l}{E-M} = |N|^2 \frac{2p^i}{E-M} \Rightarrow \vec{J} = |N|^2 \frac{2\vec{p}}{E-M}$$

Hence, the 4-current can be written as:

$$J^\mu = \bar{\Psi} \gamma^\mu \Psi = |N|^2 \left(\frac{2E}{E-M} ; \frac{2\vec{p}}{E-M} \right)$$

It can be shown and it was done in class that the normalization $|N|^2 = |E| + M$
Therefore,

$$J^\mu = \bar{\Psi} \gamma^\mu \Psi = |N|^2 \left(\frac{2E}{E-M} ; \frac{2\vec{p}}{E-M} \right) = (|E|+M) \left(\frac{2E}{-|E|-M} ; \frac{2\vec{p}}{-|E|-M} \right) \Rightarrow$$

$$J^\mu = (-2E ; -2\vec{p}) = -2p^\mu = 2(|E| ; -\vec{p}) \qquad E < 0$$

This result should be compared with that from positive energy solutions. As we have shown before the current for positive energy solutions is:

$$J^\mu = \bar{\Psi} \gamma^\mu \Psi = |N|^2 \left(\frac{2E}{E+M} ; \frac{2\vec{p}}{E+M} \right) \text{ and } |N|^2 = E + M$$

Which gives: $\qquad J^\mu = +2p^\mu$

Problem 4: 1. Show that the helicity operator commutes with the Hamiltonian:

$$[\vec{\Sigma} \cdot \hat{p}, H] = 0$$

2. Show explicitly that the solutions of the Dirac equation are eigenvectors of the helicity operator:

$$\vec{\Sigma} \cdot \hat{p} \;\; \Psi = \pm \Psi$$

Solution:

1. We have that:

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \text{ and } \vec{\Sigma}\cdot\hat{p} = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}$$

$$\left[H, \vec{\Sigma}\cdot\hat{p}\right] = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} - \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \Rightarrow$$

$$\left[H, \vec{\Sigma}\cdot\hat{p}\right] = \begin{pmatrix} m\vec{\sigma}\cdot\hat{p} & \vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\hat{p} \\ \vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\hat{p} & -m\vec{\sigma}\cdot\hat{p} \end{pmatrix} - \begin{pmatrix} m\vec{\sigma}\cdot\hat{p} & \vec{\sigma}\cdot\hat{p}\,\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\hat{p}\,\vec{\sigma}\cdot\vec{p} & -m\vec{\sigma}\cdot\hat{p} \end{pmatrix} = 0 \Rightarrow$$

$$\left[H, \vec{\Sigma}\cdot\hat{p}\right] = 0$$

2. We will show this for the negative energy solutions. For positive solutions it works the same way:

$$\vec{\Sigma}\cdot\hat{p}\,\Psi^{(-)}(x) = \sqrt{(|E|+M)}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ 1 \end{pmatrix}\chi^s e^{-ipx} \Rightarrow$$

$$\vec{\Sigma}\cdot\hat{p}\,\Psi^{(-)}(x) = \sqrt{(|E|+M)}\begin{pmatrix} \vec{\sigma}\cdot\hat{p}\dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ \vec{\sigma}\cdot\hat{p} \end{pmatrix}\chi^s e^{-ipx} \Rightarrow$$

$$\vec{\Sigma}\cdot\hat{p}\,\Psi^{(-)}(x) = \sqrt{(|E|+M)}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ 1 \end{pmatrix}(\vec{\sigma}\cdot\hat{p})\chi^s e^{-ipx} \tag{A}$$

However we can always choose a the momentum vector on the z-axis where:

$$(\vec{\sigma}\cdot\hat{p})\chi^s = \pm\chi^s \tag{B}$$

Hence from (A) and (B) we get that:

$$\vec{\Sigma}\cdot\hat{p}\,\Psi^{(-)}(x) = \pm\sqrt{(|E|+M)}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E-M} \\ 1 \end{pmatrix}\chi^s e^{-ipx} = \pm\Psi^{(-)}(x)$$