Particle Physics Homework Assignment 5

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Problem 1: As shown in class the Dirac matrices must satisfy the anti-commutator relationships:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}\,, \quad \{\alpha_i, \beta\} = 0 \quad \text{with} \quad \beta^2 = 1$$

I. Show that the α_i , β are Hermitian, traceless matrices with eigenvalues ± 1 and even dimensionality.

II. Show that, as long as the mass term is not zero and the matrix β is needed, there is no 2x2 set of matrices that satisfy all the above relationships. Hence, the Dirac matrices must be of dimension 4 or higher. First show that the set of matrices $(1;\vec{\sigma})$ can be used to express any 2×2 matrix. That is the coefficients c_0, c_i always exist such that any 2×2 matrix can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = c_0 \cdot 1 + c_i \cdot \sigma_i$$

Having shown this you can pick and intelligent choice for the α_i in terms of the Pauli matrices, for example $\alpha_i = \sigma_i$ which automatically obeys $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$, and express β in terms of $(1;\vec{\sigma})$ using (1) . Show then that there is no 2×2 β matrix that satisfies $\{\alpha_i, \beta\} = 0$.

Solution:

(a) The Dirac Hamiltonian is given by $H = \vec{\alpha}\cdot\vec{p} + \beta m$. Hence, α_i , β are required to be hermitian and as one can see in the standard representation the are indeed Hermitian.

(b) $$\alpha_i \Psi = \lambda \Psi \Rightarrow \alpha_i^2 \Psi = \lambda^2 \Psi \qquad (1)$$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \Rightarrow 2\,\alpha_i^2 = 2\delta^{ii} \Rightarrow \alpha_i^2 = 1 \qquad (2)$$

where in (2) the double index does not imply summation in this case.

From (1) and (2) we have that: $\lambda = \pm 1$. Similarly for β we have that

$$\beta\Psi = \lambda\Psi \Rightarrow \beta^2\Psi = \lambda^2\Psi \Rightarrow \Psi = \lambda^2\Psi \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Furthermore

$$\{\alpha_i, \beta\} = 0 \Rightarrow \alpha_i \beta + \beta \alpha_i = 0 \Rightarrow \alpha_i = -\beta \alpha_i \beta \qquad (3)$$

Hence using (3) and $\beta^2 = 1$ we have that:

$$Tr(\alpha_i) = Tr(-\beta \alpha_i \beta) = -Tr(\beta^2 \alpha_i) = -Tr(\alpha_i) \Rightarrow Tr(\alpha_i) = 0$$

(recall that $tr(A \cdot B) = tr(B \cdot A)$)

Similarly we have that:

$$\{\alpha_i, \beta\} = 0 \Rightarrow \beta = -\alpha_i \beta \alpha_i \Rightarrow Tr(\beta) = -Tr(\alpha_i \beta \alpha_i) \Rightarrow$$

$$Tr(\beta) = -Tr(\beta) \Rightarrow Tr(\beta) = 0$$

Since the Dirac matrices have eigenvalues +/- 1 and are traceless they must have even-dimensionality. This is because they are hermitian, one can diagonalise them and in this case the diagonal elements are the eigenvalues. So the sum of the eigenvalues (trace) must be zero. If the eigenvalues are only +/- 1 the only way this can happen is if there is an equal number of +1 and -1 (even dimensionality)

(b) Any 2x2 matrix can be written as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c_0 \cdot 1 + c_i \cdot \sigma^i$$

because for every set of a, b, c, d there is always a set of c_i such that:

$$a = c_0 + c_3 \; ; \; d = c_0 - c_3 \; ; \; b = c_1 - ic_2 \; ; \; c = c_1 + ic_2 \Rightarrow$$

$$c_0 = \left(\frac{(a+d)}{2}\right) \; ; \; c_3 = \left(\frac{(a-d)}{2}\right) \; ; \; c_1 = \left(\frac{(b+c)}{2}\right) \; ; \; c_2 = \left(\frac{(c-b)}{(2i)}\right)$$

Now suppose that : $\beta = c_0 \cdot 1 + c_i \cdot \sigma^i$ and $\alpha^i = \sigma^i$ if this set of gamma matrices is to be a valid one it should satisfy the Dirac commutation relationships. Hence,

$$\{c_0 \cdot 1 + c_i \cdot \sigma^i, \sigma^j\} = 0 = c_0 2\sigma^j + c_i\{\sigma^i, \sigma^j\} = 2c_0\sigma^j + 2c_i\delta^{ij} = 2c_0\sigma^j + 2c_j$$

The only way that this last expression can be zero is if both coefficients are always zero. Hence, there is no non-trivial choice for **β** and a 2×2 representation of the Dirac matrices is not possible. As we will see later, if the particle is massless then we don't need the **β** matrix and then a 2×2 representation is possible. The solutions of such an equation will be then two component spinors which in the past were the basis for the two-component neutrino theory (in the mean time we know that neutrinos do have mass albeit very small).

Problem 2:

1. Show that the Weyl matrices:

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfy all the Dirac conditions of Problem 1. Hence, they form just another representation of the Dirac matrices, the Weyl representation, which is different than the standard Pauli-Dirac representation.

2. Show the the Dirac matrices in the Weyl representation are

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3. Show that in the Weyl representation $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

1. $$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \Rightarrow \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = 2\delta^{ij}\cdot 1$$

$$\Rightarrow \begin{pmatrix} +\sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} +\sigma^j\sigma^i & 0 \\ 0 & \sigma^j\sigma^i \end{pmatrix} = 2\delta^{ij}\cdot 1$$

$$\begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} = 2\delta^{ij}\cdot 1 \Rightarrow$$

$$\begin{pmatrix} \{\sigma^i,\sigma^j\} & 0 \\ 0 & \{\sigma^i,\sigma^j\} \end{pmatrix} = 2\delta^{ij}\cdot 1 \Rightarrow$$

$$\begin{pmatrix} 2\delta^{ij} & 0 \\ 0 & 2\delta^{ij} \end{pmatrix} = 2\delta^{ij}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\delta^{ij}\cdot 1 \Rightarrow$$

$$\{\alpha_i,\beta\} = 0 \Rightarrow \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = 0$$

$$\beta^2 = 1 \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So the Weyl representation is just another representation which satisfies the same anti-commutation relationships as the Pauli-Dirac (standard) representation.

2. Just as in the Dirac Pauli case one can define the Dirac gamma matrices as:

$$\vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\gamma^0 = \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since we have already proven that the $\vec{\alpha},\beta$ satisfy the Dirac anti-commutation relationships, it can be shown easily as done in the standard representation that the gamma matrices of the Weyl representation also satisfy:

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$$

3. Again we have by definition that $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Hence,

$$\gamma_5 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \Rightarrow$$

$$\gamma_5 = i\begin{pmatrix} -\gamma^1 & 0 \\ 0 & \gamma^1 \end{pmatrix}\begin{pmatrix} 0 & \gamma^2 \\ -\gamma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \Rightarrow$$

$$\gamma_5 = i\begin{pmatrix} 0 & -\sigma^1\sigma^2 \\ -\sigma^1\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \Rightarrow$$

$$\gamma_5 = i\begin{pmatrix} 0 & -i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \Rightarrow$$

$$\gamma_5 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Problem 3: Use the Dirac Hamiltonian in the standard Pauli-Dirac representation,

$$H = \vec{\alpha}\cdot\vec{p} + \beta m$$

to compute $[H, \hat{L}]$ and $[H, \hat{\Sigma}]$ and show that they are not zero.
Use the results to show that:

$$[H, \hat{L} + (\tfrac{1}{2})\hat{\Sigma}] = 0$$

where the components of the angular momentum operator is given by:

$$\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$$

and the components of the spin operator are given by:

$$\hat{\Sigma}_i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

Recall that the Pauli matrices satisfy $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

Solution: In the Dirac Pauli representation we have that:

$$H = \vec{\alpha}\cdot\vec{p} + \beta m = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \qquad (1)$$

and

$$[H, L_j] = [\alpha_i p_i, L_j] = \alpha_i[p_i, \varepsilon_{jlm} x_l p_m] = \alpha_i \varepsilon_{jlm} x_l [p_i, p_m] + \alpha_i \epsilon_{jlm}[p_i, x_l] p_m \Rightarrow$$

$$[H, L_j] = \alpha_i \varepsilon_{jlm} x_l 0 + \alpha_i \varepsilon_{jlm}(-i\delta_{il}) p_m = -i\varepsilon_{jlm}\alpha_l p_m = -i(\vec{\alpha}\times\vec{p})_j \Rightarrow$$

$$[\hat{H},\vec{L}] = -i(\vec{\alpha}\times\vec{p}) \tag{2}$$

(all the variables are QM operators and obey QM commutation relationships)

$$[H,\hat{\Sigma}] = [\vec{\alpha}\cdot\vec{p}+\beta m,\vec{\Sigma}] = \left[\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix},\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\right] \Rightarrow$$

$$[H,\hat{\Sigma}] = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}-\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \Rightarrow$$

$$[H,\hat{\Sigma}] = \begin{pmatrix} 0 & [\vec{\sigma}\cdot\vec{p},\vec{\sigma}] \\ [\vec{\sigma}\cdot\vec{p},\vec{\sigma}] & 0 \end{pmatrix} \tag{3}$$

However: $\sigma^i\sigma^j = \delta^{ij}+i\varepsilon^{ijk}\sigma^k \Rightarrow [\sigma^i,\sigma^j] = 2i\varepsilon^{ijk}\sigma^k$ (4)

Using (3) and (4) we get:

$$[H,\vec{\Sigma}]_j = \begin{pmatrix} 0 & [\sigma^i p^i,\sigma^j] \\ [\sigma^i p^i,\sigma^j] & 0 \end{pmatrix} = 2i\varepsilon^{ijk}\begin{pmatrix} 0 & p^i\sigma^k \\ p^i\sigma^k & 0 \end{pmatrix} \Rightarrow$$

$$[H,\hat{\Sigma}]_j = 2i\varepsilon^{ijk}\begin{pmatrix} 0 & p^i\sigma^k \\ p^i\sigma^k & 0 \end{pmatrix} = -2i\varepsilon^{jik}p^i\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = -2i\varepsilon^{jik}p^i\alpha^k \Rightarrow$$

$$[H,\hat{\Sigma}]_j = -2i\vec{p}\times\vec{\alpha} \tag{5}$$

and from (3) and (5) we have that:

$$[H,\hat{L}+(\frac{1}{2})\hat{\Sigma}] = -i(\vec{a}\times\vec{p})+(\frac{1}{2})(-2i\vec{p}\times\vec{\alpha}) = 0$$

Problem 4: Show that $(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0$

Solution:

The proof of this is based on the fact that the matrixes a_i and β are hermitian.

a) For γ^0 it is rather obvious $(\gamma^0)^+ = \beta^+ = \gamma^0\gamma^0\gamma^0 = \beta\beta\beta = \beta$ which is true because the matrix β is iindeed hermitian.

b) $(\gamma^i)^+ = (\beta\alpha^i)^+ = (\alpha^i)^+(\beta)^+ = \alpha^i\beta = -\beta\alpha^i = -\gamma^i = \gamma^0\gamma^i\gamma^0$

alternatively

$(\alpha_i)^+ = \alpha_i \Rightarrow (\beta\gamma^i)^+ = \beta\gamma^i \Rightarrow (\gamma^i)^+\beta^+ = \beta\gamma^i \Rightarrow (\gamma^i)^+\beta = \beta\gamma^i \Rightarrow$

$(\gamma^i)^+ = \beta\gamma^i\beta$. Hence,

$$(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0$$