Particle Physics Homework Assignment 3

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Problem 1: The HERA accelerator, which operated at the DESY laboratory in Hamburg Germany in the period between 1992 and 2007, collided 27.5 GeV electrons with 920 GeV protons.

I. Compute the centre of mass energy (total available energy at the electron proton centre of mass) assuming that the angle between the proton and the electron beam momenta is 180^0 (head on collision).

II. Compute the boost, $\vec{\beta}_{CM}$, of the electron-proton centre of mass frame relative to the laboratory frame.

III. What should be the energy of an electron beam colliding with protons at rest if the centre of mass energy were to be the same with HERA ? This type of experiment is called *fixed target experiment* to distinguish it with the previous which is called a *collider experiment.*

Solution:

I. Define $P_e = (P_e^0; \vec{P}_e)$ and $P_p = (P_p^0; \vec{P}_p)$ to be the beam electron and proton 4-vectors. We already know that the centre-of-mass-energy-square is given by : $s = (P_e+P_p)^2 = m_e^2+m_P^2+2P_e \cdot P_p$. A the energies we are dealing here we can neglect the mass terms and since the beams collide head-on the 4-vector product can be simplified. Hence,

$$s = 4P_e^0 P_p^0 = 101200\, GeV^2 \Rightarrow \sqrt{s} \approx 318 \;\; GeV$$

II. The boost is given by: $$\vec{\beta} = \frac{\vec{P}_p+\vec{P}_e}{P_e^0+P_p^0} = \frac{920\,\hat{x}-27.5\,\hat{x}}{920+27.5} = 0.942\,\hat{x}$$

III. For fixed target we have: $P_e = (P_e^0; \vec{P}_e)$ and $P_p = (m_p; \vec{0})$. Hence,

$$s = (P_e+P_p)^2 = m_e^2+m_P^2+2P_e^0 \cdot m_p$$

we neglect again the electron and proton masses and we get:

$$s = 2P_e^0 m_p$$

Therefore, $\mathbf{101200}\ \boldsymbol{GeV^2} = \mathbf{2}\,\boldsymbol{P_e^0 \cdot m_p} \Rightarrow \boldsymbol{P_e^0}=\mathbf{50600}\ \boldsymbol{GeV} = \mathbf{50.6}\,\boldsymbol{TeV}$ where the proton mass has been approximately set to 1 GeV. As seen here to achive the same centre-of-mass-energy at a fixed target experiment one would have to produce a beam of significantly higher energy and this is the reason that we use colliders to reach high centre-of-mass-energies.

Problem 2 : Deduce an expression for the energy of γ-rays from the decay of the neutral pion, $\pi^0 \rightarrow \gamma\gamma$, in terms of the mass **m**, energy **E**, velocity βc of the pion and the angle of emission θ^* of the photon in the pion rest frame. Because the pions have zero spin the angular distribution is isotropic at the pion rest frame. Show that the γ-ray energy spectrum in the laboratory frame will be flat extending from $E(1+\beta)/2$ to $E(1-\beta)/2$. For relativistic pions, find an expression for the disparity D of the γ-rays and show that for D>3 one observes half the decays and for D>7 one quarter of them. (Perkins p33, 4th edition). The disparity D is defined as the ratio of the energy of the most energetic photon divided by the energy of the least energetic photon.

Solution:

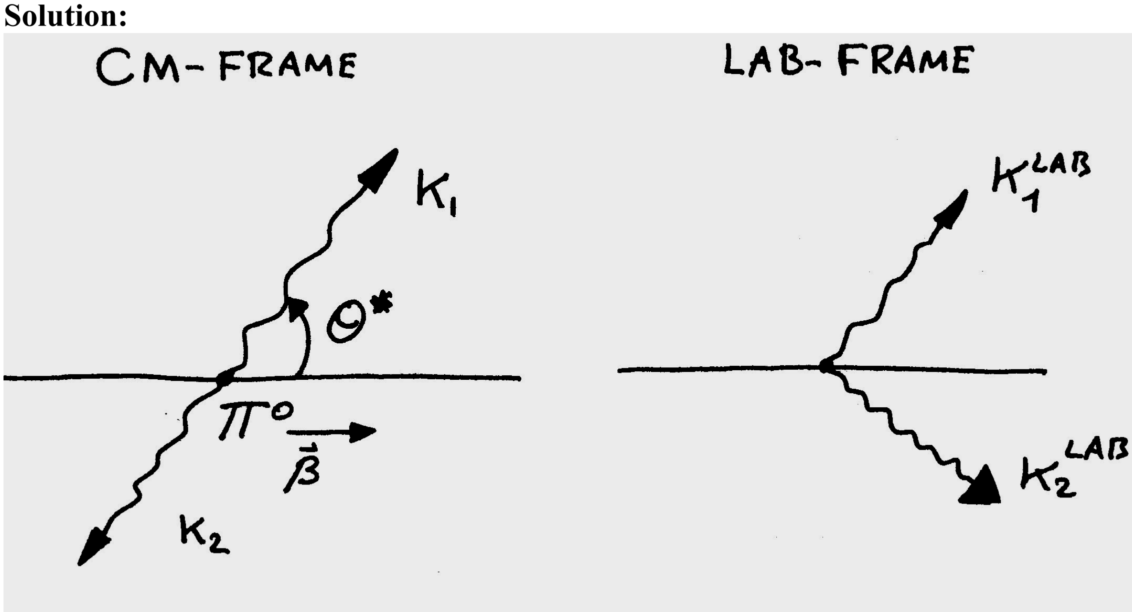

Figure 1: A π^0 moving in the positive x-direction and decaying in the centre of mass frame (left) and the laboratory frame (right).

In the CM system we have $k_1=(k_1^0;\vec{k}_1)$ and $k_2=(k_2^0;\vec{k}_2)$. However because it is the CM system $\vec{k}_1=-\vec{k}_2$. The two outgoing particles are photons and have zero mass. Hence, $\Rightarrow\ k_1^0 = k_2^0 = |\vec{k}_1| = |\vec{k}_2|$. Using then energy conservation we have that

$$m_{\pi^0} = k_2^0 + k_1^0 = 2\cdot k_1^0 \quad \Rightarrow \quad k_1^0 = \frac{m_\pi^0}{2}$$

And the two 4-vectors in the CM frame are completely determined from the kinematics:

$$k_1=\frac{m_{\pi^0}}{2}(1,\vec{n})\,, \quad k_2=\frac{m_{\pi^0}}{2}(1,-\vec{n}) \qquad (1)$$

The photon energy in the laboratory frame is given by:

$$k_1^{LAB}=(E_1^{LAB};\vec{p}_1^{\,LAB})$$

and can be computed using the Lorentz transformation:

$$E_1^{LAB} = \gamma(k_1^0+\vec{\beta}\cdot\vec{k}_1) \qquad (2)$$

$$(1)(2) \qquad \Rightarrow E_1^{LAB} = \frac{m_{\pi^0}\gamma}{2}(1+\beta\, cos\theta^*) \qquad (3)$$

where γ, $\vec{\beta}$ refer to the pion boost in the laboratory frame with:

$$E_{\pi^0}^{LAB} = \gamma m_{\pi^0} \qquad (4)$$

$$(3)(4) \qquad \Rightarrow E_1^{LAB} = \frac{E_{\pi^0}^{LAB}}{2}(1+\beta\, cos\theta^*) \qquad (5)$$

The spin of the pion is of course zero and this means that there is no prefered direction at the pion rest frame. Therefore the decay distribution in the pion rest frame will be isotropic. In other words the number of photons produced per solid angle will be a constant, C:

$$\frac{dN}{d\,\Omega}=C \;\Rightarrow\; \frac{dN}{dcos\theta^*\, d\,\varphi^*} = C \;\Rightarrow\; \frac{dN}{dcos\theta^*} = 2\pi C = Constant$$

$$(5) \quad \Rightarrow \frac{dN}{dE_1^{LAB}} = \frac{2}{E_{\pi^0}^{LAB}\,\beta} \times \frac{dN}{d\cos\theta^*} = Flat$$

The limits of the distribution can also be derived with the help of (5). The photons emitted at the direction of the pion have the maximum energy equal to:

$$(5) \quad \Rightarrow E_1^{LAB-MAX} = \frac{E_{\pi^0}^{LAB}}{2}(1+\beta)$$

while those emitted opposite to the direction of the pion will have the minimum energy:

$$(5) \quad \Rightarrow E_1^{LAB-MAX} = \frac{E_{\pi^0}^{LAB}}{2}(1-\beta)$$

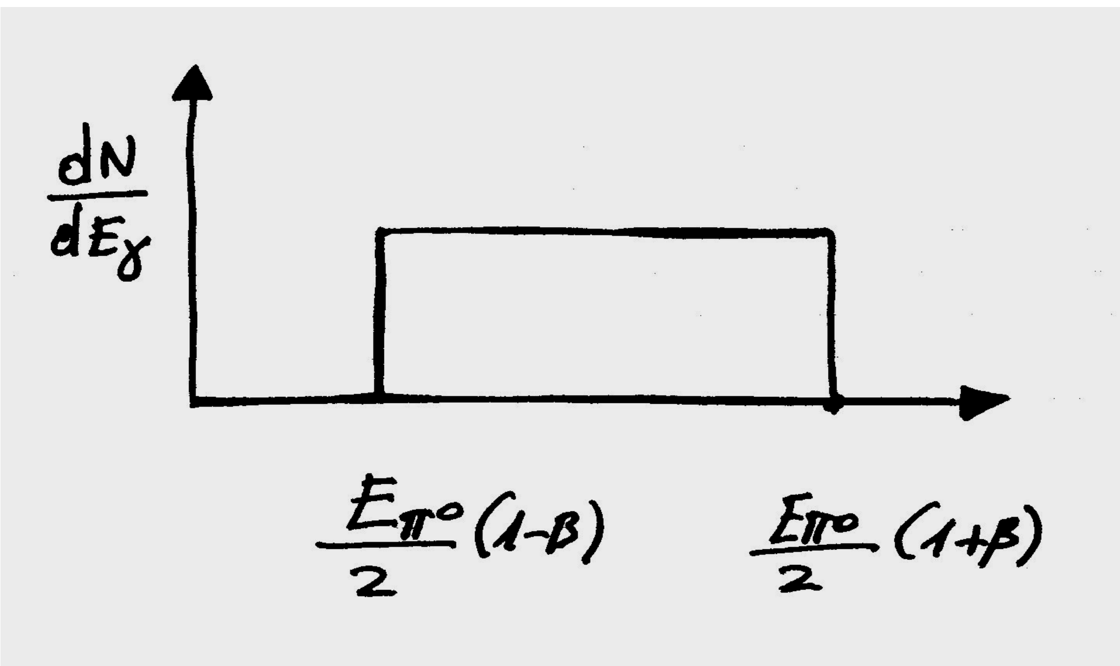

Figure 2: The photon distribution in the Laboratory frame.

The energy disparity, **D**, of the two photons is given by:

$$D = \frac{E_1^{LAB}}{E_2^{LAB}} = \frac{1+\beta\cos\theta^*}{1-\beta\cos\theta^*} \approx \frac{1+\cos\theta^*}{1-\cos\theta^*} \qquad (6)$$

Note that the disparity distribution has a range from 1 to infinity be definition since for each event we divide always the most energetic photon by the least energetic one. Double counting is avoided by not allowing the $\cos\theta^*$ to become negative.

In order to compute the population of decays above a certain value of disparity, $\mathbf{D_0}$, one needs to know the disparity distribution $\frac{dN}{dD}$ and integrate from $\mathbf{D_0}$ to infinity.

$$\frac{dN}{dD} = \frac{dN}{d\cos\theta^*} \times \frac{d\cos\theta^*}{dD} \tag{7}$$

The first term in **(6)** known to be a constant and the second we need to compute:

Let
$$\cos\theta^* = x \tag{8}$$

$$(6)(8) \Rightarrow \frac{dD}{dx} = \frac{2}{(1-x)^2} \tag{9}$$

$$(6)(8) \Rightarrow x = \frac{D-1}{D+1} \tag{10}$$

and using (7) we get:
$$\frac{dN}{dD} = \frac{dN}{d\cos\theta^*} \times \frac{2}{(1+D)^2} \Rightarrow$$

$$N(D>D_0) = \int \frac{dN}{dD} dD = \frac{dN}{d\cos\theta^*} \times \int \frac{2}{(1+D)^2} dD \Rightarrow$$

$$N(D>D_0) = \frac{dN}{d\cos\theta^*} \times \frac{2}{(1+D_0)}$$

Hence:

$$N(D>3) = \frac{dN}{d\cos\theta^*} \times \frac{2}{(1+3)} = \frac{1}{2} \times \frac{dN}{d\cos\theta^*}$$

$$N(D>7) = \frac{dN}{d\cos\theta^*} \times \frac{2}{(1+7)} = \frac{1}{4} \times \frac{dN}{d\cos\theta^*}$$

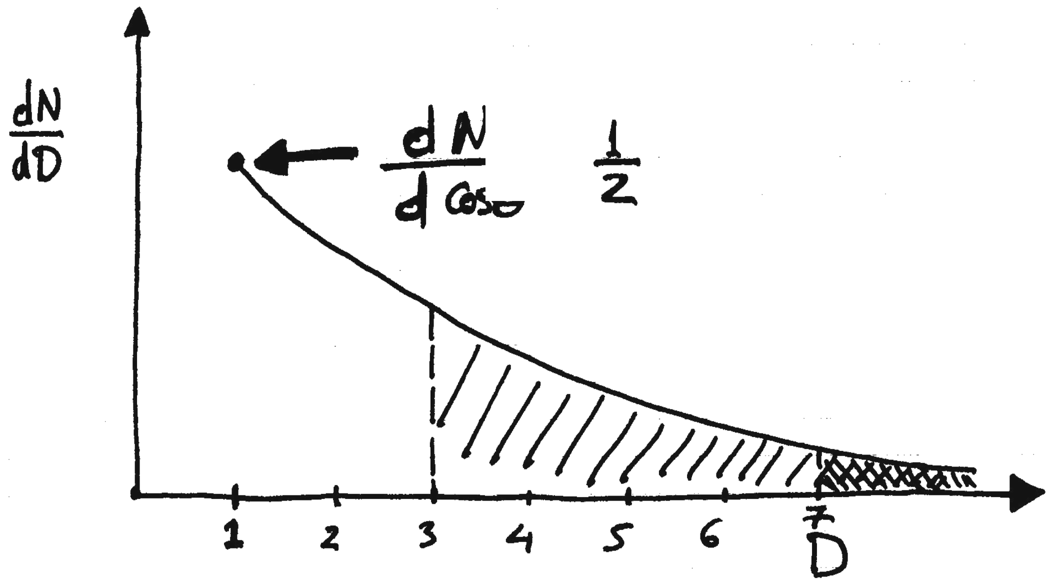

Figure 3: The Energy disparity distribution in the laboratory frame.

Problem 3: A high energy electron collides with an atomic electron which can be considered at rest. What is the threshold (the minimum kinetic energy of the incident electron) for producing and electron positron pair ?

Solution:

We are asked to find out what is the minimum kinetic energy so that the reaction

$$e^- \ e^- \rightarrow e^- \ e^- \ e^- \ e^+$$

occurs. In other words what is the minimum kinetic energy of the incident electron so that the 4 outgoing particles are produced with zero kinetic energy (the only energy they have is due to rest mass).

Let $P_1^\mu = (E;\vec{p})$ be the 4-vector of the incident electron, $P_2^\mu = (m_e;\vec{0})$ that of the atomic electron assumed at rest, and $P_3^\mu = P_4^\mu = P_5^\mu = P_6^\mu = (m_e;\vec{0})$ the 4 vectors of the final state particles.

Conservation of energy and momentum during the collision dictates that:

$$P_1^\mu + P_2^\mu = P_3^\mu + P_4^\mu + P_5^\mu + P_6^\mu \qquad \Rightarrow$$

$$(P_1 + P_2)^2 = (P_3 + P_4 + P_5 + P_6)^2 \Rightarrow$$

$$2m_e^2 + 2E\,m_e = 4m_e^2 + 2\sum_{2<j<i<6} p_i p_j \Rightarrow$$

$$2m_e^2 + 2E\,m_e = 4m_e^2 + 2\cdot 6\,m_e^2 = 16\cdot m_e^2 \Rightarrow$$

$$E = 7\cdot m_e$$

So the kinetic energy of the incident electron must be at least 6 times the mass of the electron for an extra electron positron to be produced. This is precisely the way that the anti-poton was discovered via the reaction

$$p + p \to p + p + p + \bar{p}$$

in 1955 at Berkley and this is why the accelerator was designed to produce a proton beam up to 6.2 GeV. At the time the unit GeV was called BeV. Hence the accelerator was called **Bevatron**. The results of this experiment have been published in: O. Chamberlain, E. Segre, C Wiegand, T. Ypsilantis, Phys. Rev. 100, 947, (1955). Also another experiment at the Bevatron discovered the antineutron: B. Cork, G. R Lambertson, O. Piccioni, W. A.Wenzel, Physical Review. **104**: 1193–1197 (1956).
doi:10.1103/PhysRev.104.1193

Problem 4: An experiment which measured parity violation in weak interactions as well as the magnetic moment and the Lande' g factor of the μ^+, used a μ^+ beam which was produced from the decay in flight of an incident π^+ beam ($\pi^+ \rightarrow \mu^+ \ \nu_\mu$). The kinetic energy of the π^+ beam was **85 MeV**. The π^+ beam was produced by colliding a proton beam from an accelerator with a target and selecting the positively charged π^+ from the negatively charged π^- using a magnetic field. Positive pions of a certain direction and momentum were selected by passing the π^+ beam through an appropriately positioned concrete block with a straight hole through it.

Questions regarding the beam:

I. Compute the decay length of the pions in the laboratory frame. The lifetime of the π^+ is **26 ns** and the π^+ mass is about **140 MeV**. The incident beam to the experiment was a mixture of 10% μ^+ and 90% π^+. How far was the experiment from the point where the pions were produced ?

II. Compute the pion mean free path in carbon assuming that the pion carbon cross section at the relevant energy is **10 mb**. This cross section includes all strong interaction processes that contribute. The carbon density is **ρ = 2.265 g/cm³**.

III. Compute the muon and neutrino energies in the rest frame of the pion. Assume that the neutrino is massless and that the mass of the muon is $m_\mu = 106\, MeV$.

IV. Compute the maximum and minimum energy of the muon in the lab frame.

V. The authors of the paper placed a carbon block, approximately **20 cm** long, infront of a tileted carbon target. Justify the need for this block given that pions and muons of this energy lose in carbon approximately **4.5 MeV/cm** due to ionization. Energy loss via ionization is the topic of the next lecture.

Questions Regarding the Apparatus:

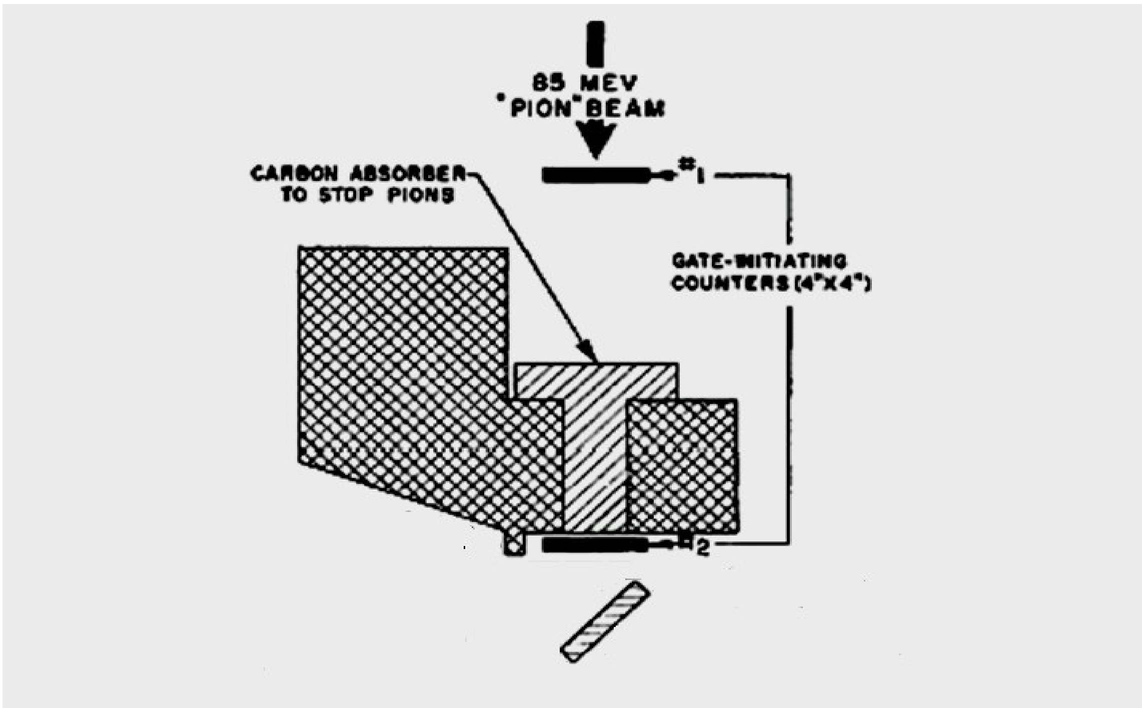

Figure 4: Part of the apparatus of the Garwin-Lederman-Weinrich experiment Phys. Rev.,105:1415-1417, 1957. The 85 MeV pion beam at Columbia University NEVIS labs is shown entering from the top. The carbon absorber length has been chosen so that only muons exit the carbon block whilst the pions stop in the block. Hence a signal coincidence from the counters #1 and #2 indicates that a muon has gone through. If the carbon length is chosen appropriately most muons will likely stop on the tilted carbon target below.

The experimental apparatus is shown in Fig. 1. A carbon block has been used to separate the pions from the muons. Actually the carbon block was introduced to for two reasons: (1) to stop the pions (2) to slow down the muons in such a way so that when they exit the carbon block they have very little kinetic energy left and thus they stop at the carbon target shown tilted at the bottom of Fig. 4. There they decay giving one electron and two neutrinos and the properties of the decay electron can be measured.

Two counters #1 and #2 have been placed at the path of the beam before and after the carbon block. Counters are devices made of liquid or plastic scintillator. When a charged particle goes through a counter it excites the scintillator material and produces light which can be collected and converted by photo-tubes to charge. Detection of this charge using electronic circuits provides information which indicates that a particle has gone through the scintillator material as well as allows a measurement of the time that the particle went through.

The signals from Counters #1 and #2 in this experiment are used in coincidence to provide an experimental trigger. This means that if both counters detected that a particle went though at the same time[1] (coincidence trigger) then that meant that a muon went through the carbon and exited from the other side albeit with very little kinetic energy so that it most likely stopped on the target. If the trigger coincidence condition was fulfilled then detectors around the target were activated appropriately and recorded the data from the muon decays.

VI. Compute the maximum possible angle of the decay muons from pions in the Laboratory frame. Use the results from (V) and explain how does the carbon block constrain this angle. Do the angles of the muons which are capable to penetrate the block extend up to this angle ?

VII. Given that this experiment does not record data unless if *there is a trigger,* compute and discuss how does the distance between the two counters affects the muon energy spectrum detected in this experiment. Assume that the beam enters at the center of the fist counter.

Hence, this part of the apparatus has been designed to collect muons which stop at the target (for sometime since the muon lives only 2.16 μsec) to study their properties. The rest of this experiment as well as the results of it will be discussed later in the course when we discuss parity violation in weak interactions and and about the muon magnetic moment.

Solution:

I. The kinetic energy of the pions is **KE = 85 MeV**. Hence,

$$E_\pi = 85+140\,MeV = 225\,MeV \Rightarrow$$

$$\gamma = \frac{E_\pi}{m_\pi} = \frac{225\,MeV}{140\,MeV} = 1.61 \Rightarrow \beta = 0.783\,; \;\; \beta\gamma = 1.26$$

$$l = c\beta\gamma\tau = 1.61 \times 0.783 \times 3 \times 10^{8} \times 26 \times 10^{-9}\,m = 9.83\,m$$

If the beam contains only 10% muons this means that 90% of the pions have survived at the entry point of the experiment which we assume it to be located at distance D from the pion production point.

[1] The time difference introduced by the muon travelling between the two counters is very small and the experiment is not sensitive to it.

$$\frac{N_I}{N_0} = e^{\frac{-t}{(\gamma\tau)}} \Rightarrow \ln\left(\frac{N_I}{N_0}\right) = (-1)\times\frac{t}{\gamma\tau} \Rightarrow -.105 = (-1)\times\frac{D}{c\,\beta\gamma\,\tau} \Rightarrow D = 1.03\,m$$

II.

$$\lambda = \frac{A}{N_A \times \rho \times \sigma} = \frac{12.01\,gr\,mole^{-1}}{6.022\times10^{23}\,atoms\,mole^{-1}\times2.265\,gr\,cm^{-3}\times10\times10^{-3}\times10^{-28}\,m^2}$$

$$\lambda = 880\,cm$$

Hence, strong interaction processes which contribute the pion-nucleon cross section (10 mb) do not result to enough energy loss to stop the pions in the 20 cm of carbon. However, the pions lose energy also via another process, the ionization, at a rate of 4.5 MeV/cm (see next lecture).

III. This is a two-body decay, $\pi^+ \rightarrow \mu^+ \ \nu_\mu$, therefore in the pion rest frame we have

$$P_\pi = P_\nu + P_\mu \Rightarrow (P_\pi - P_\nu)^2 = P_\mu^2 \Rightarrow m_\pi^2 - 2\,m_\pi E_\nu = m_\mu^2 \text{ and finally}$$

$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2\,m_\pi} = \frac{140^2 - 106^2}{2\times140}\,MeV = 29.87\,MeV \simeq 30\,MeV$$

Assuming that the neutrino is massless we have that $p_\nu = E_\nu$ in units of $\hbar = c = 1$ (perhaps now you start appreciating the use of these units). At the pion rest frame (two body decay) we have that:

$$p_\mu = p_\nu = E_\nu \qquad (1)$$

Hence $$E_\mu = \sqrt{E_\nu^2 + m_\mu^2} = 110.1\,MeV \simeq 110\,MeV \qquad (2)$$

IV. Muons of maximum energy at the laboratory frame are those that travel at the pion direction in the pion frame and those that travel opposite to it will have the minimum energy in the lab frame. This statement makes perfect sense but can also be derived from the Lorentz transformation:

$$E_\mu^{LAB} = \gamma(E_\mu + \vec{\beta}\,\vec{p}_\mu) \qquad (A)$$

and using (A) $$E_\mu^{LAB-MAX} = 1.61(110 + 0.783\cdot30)\,MeV = 215\,MeV$$

$$E_{\mu}^{LAB-MIN} = 1.61(110-0.783\times 30)MeV = 139\,MeV$$

V. Easy: $L = \frac{85}{4.5}cm \approx 19\,cm$ so **20 cm** is enough so that no pions exit. They will stop in the carbon block and decay there. However the kinetic energy of the muons from the pion decay in flight at the laboratory system is between

$$139-106 \leqslant KE_{\mu}^{LAB} \leqslant 215-106 \quad MeV \quad \Rightarrow \quad 33\,MeV \leqslant KE_{\mu}^{LAB} \leqslant 109 \quad MeV$$

So with 20 cm carbon the muons will lose at most $4.5\frac{MeV}{cm}\times 20\,cm = 90\,MeV$ so not all of them will come out of the block. Muons with kinetic energy above **90 MeV** will definitely exit the carbon block and will have kinetic energies in the range between **0-19 MeV.** If their remaining KE is sufficiently low they will stop at the carbon target and decay there.

VI. Let θ, p^{μ} be the angle and the 4-momentum of the decay muon in the pion frame and $\theta_{LAB}, p^{\mu}_{LAB}$ be the angle and 4-momentum of the muon in the laboratory frame. The Lorentz transformation from the the pion to the laboratory frame gives:

$$p^0_{LAB} = \gamma(p^0+\vec{\beta}\cdot\vec{p}) = \gamma\, p^0+\beta\gamma\, p\cos\theta \tag{3}$$

$$p^1_{LAB} = \gamma\, p^1+\beta\gamma\, p^0 = \gamma\, p\cos\theta+\beta\gamma\, p^0 \tag{4}$$

$$p^2_{LAB} = p^2 = p\sin\theta \tag{5}$$

(The axis with index 1 is along the the boost direction and the axis with index 2 is perpendicular to it).

$$\tan(\theta_{LAB}) = \frac{p^2_{LAB}}{P^1_{LAB}} = \frac{p\ \sin\theta}{\gamma\, p\cos\theta+\beta\gamma\, p^0} = \frac{\sin\theta}{\gamma\cos\theta+\beta\gamma\frac{p^0}{p}}$$

Demand that $\tan(\theta_{LAB})$ has a maximum as a function of θ and you get that

$$\cos\theta^{MAX} = (-1)\frac{p}{\beta\, p^0}=\frac{-30}{0.783\times 110} \quad \Rightarrow \quad \theta^{MAX} = 110^0 \text{ in the pion frame}$$

and in the laboratory frame

$$\tan\theta_{LAB} = 0.224 \Rightarrow \theta^{LAB} \leqslant 12.6^0$$

The muons capable of penetrating the block will have energies above **E_{min}=106+90 MeV.**

Hence, (3) $\Rightarrow p^0_{LAB} = \gamma p^0 + \beta\gamma p\cos\theta \geqslant E_{min} \Rightarrow$

$$cos\theta \geqslant \frac{E_{min} - \gamma p^0}{\beta\gamma p} = \frac{196 - 1.61 \times 110}{1.26 \times 30} = 0.5 \Rightarrow \theta \leqslant 60^0 \Rightarrow$$

$$\tan\theta_{LAB} \leqslant \frac{sin60^0}{1.61 \times cos60^0 + \frac{1.26 \times 110}{30}} = 0.16 \Rightarrow \theta_{LAB} \leqslant 9.1^0$$

Hence, the muons that can penetrate the block can never be emitted at angles larger than 9.1 degrees (these have energies between 196 and 215 MeV or kinetic energies between 90 and 109 MeV. It is also easy to show that the most energetic muons, those with E-215 MeV, will exit at zero angles as expected.

$$cos\theta \geqslant \frac{E_{max} - \gamma p^0}{\beta\gamma p} = \frac{210 - 1.61 \times 110}{1.26 \times 30} \approx 1 \Rightarrow \theta \approx 0^0 \Rightarrow \theta_{LAB} \approx 0^0$$

VII. By making the distance between the counters larger (i.e moving #1 further up-stream) we are constraining the muon angle in the laboratory frame as well as the angle at the pion rest frame to lower values and therefore the energy at higher values. For example muons of energy larger than 210 MeV correspond to

$$cos\theta \geqslant \frac{E_{min} - \gamma p^0}{\beta\gamma p} = \frac{210 - 1.61 \times 110}{1.26 \times 30} \Rightarrow \theta \leqslant 29.5^0 \Rightarrow \theta_{LAB} \leqslant 4.68^0$$

which can be obtained by putting the counters 61cm apart. The bottom line is that by adjusting the carbon block length and the counter distance one can determine the energy of the muons.

Problem 5: When the muon was originally discovered back in 1937 (see next lecture), people thought that this was the Yukawa particle. The Yukawa particle was considered then to be the mediator of the strong interaction. This incorrect interpretation of the nature of the muon was due to the fact that the muon mass (106 MeV) was not very different from the expected mass of the particle predicted by Yukawa. Later on it turned out that the Yukawa particle was the pion (140 MeV) which was discovered in 1947 at Bristol.

This problem[2] relates to the calculations done by Tomonaga and Araki who predicted that negative muons as they slow down in matter would be more likely to be captured by the nuclei rather than decay and the question of course is if the strong interaction is responsible for the muons which get 'swallowed' in the nucleus.

(a) Show that a negative muon captured in an S-state by a nucleus of charge Ze and mass A will spend a fraction $f \simeq 0.25\,A(Z/137)^3$ of its time inside the nuclear matter and that in time t it will travel a total distance $fct(Z/137)$ in the nuclear matter. The hydrogen atom ground state wave function can be used in these calculations with modifications to account for the fact that the muon mass is of the order of 200 times larger than the electron mass:
$\Psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{\alpha_0}\right)^{3/2} \mathrm{e}^{-\frac{Zr}{\alpha_0}}$ where $\alpha_0 = \frac{\hbar^2}{M_R e^2}$ and $M_R = \frac{A m_p m_\mu}{A m_p + m_\mu}$ is the reduced mass of the proton muon system. The proton and muon masses are $m_p = 938\,MeV$ and $m_\mu = 106\,MeV$.

(b) The law of radioactive decay of free muons is $dN/dt = -\Gamma_d N$, where $\Gamma = 1/\tau$ is the decay constant (width) and the lifetime is $\tau = 2.16\mu\,sec$. For a negative muon captured in an atom Z the decay constant is $\Gamma_{TOT} = \Gamma_d + \Gamma_c$, where Γ_c is the width for nuclear capture i.e. the probability per unit time of nuclear capture. For aluminium $(Z=13, A=27)$ the mean lifetime of negative muons is $\tau = 0.88\,\mu\,sec$. Calculate Γ_c and using he expression for f in (a), compute the interaction mean free path Λ for a muon in nuclear matter.

(c) From the magnitude of Λ estimate the magnitude of the coupling constant of the interaction that caused the nuclear capture $\mu^- + p \rightarrow n + \nu$ given that the strong interaction coupling constant is α_S and corresponds to a mean free path of $1\,fm$.

[2]This is a modified version of problem 1.9 in p34 in Perkins 4th edition.

Conversi, Pancini and Piccioni[3] did experiments in Rome in the 40s to test Tomonaga's and Araki's hypothesis and found that positive muons traversing different materials always decay rather than being captured (not surprising). They also found that negative muons undergo nuclear capture in iron rather than decay as predicted by Tomonaga and Araki[4]. Compare the mean free path for a muon to be captured by an Aluminum nucleus with the typical mean free path of a strong interaction reaction and draw conclusions as to wheather the muon could be the mediator of the strong interaction.

Solution: The fraction that the muon spends in nuclear matter is equal to the probability of the muon being in the nucleus which can be calculated using quantum mechanics:

$$f = \int_0^{r_N} \Psi^*_{100} \Psi_{100} \, dV \Rightarrow$$

$$f = \int_0^{r_N} \int_0^{\Omega} \frac{1}{\pi} \left(\frac{Z}{\alpha_0}\right)^3 e^{\frac{-2Zr}{\alpha_0}} r^2 \, dr \, d\Omega \Rightarrow$$

$$f = 4\left(\frac{Z}{\alpha_0}\right)^3 \left(\frac{\alpha_0}{2Z}\right)^3 \int_0^{\frac{2Zr_N}{\alpha_0}} e^{-y} y^2 dy \Rightarrow \tag{1}$$

However the integral can be calculated as according to:

$$f = \int_0^{y_0} y^2 e^{-y} dy = 2 - 2e^{-y_0} - 2y_0 e^{-y_0} - y_0^2 e^{-y_0} \tag{2}$$

Hence, from (1) and (2) we have that:

$$f = \left[1 - e^{-y_0} - y_0 e^{-y_0} - \left(\frac{1}{2}\right) y_0^2 e^{-y_0}\right]_{y_0 = \frac{2Zr_N}{\alpha_0}} \tag{3}$$

The quantity y_0 is a small number since it is a division of the nuclear radius by the Bohr radius. Hence, by expanding the exponentials and keeping terms up to y_0^3 we get that:

$$f = \frac{1}{6}\left(\frac{2Zr_N}{\alpha_0}\right)^3 \tag{4}$$

[3] Conversi, Pancini and Piccioni, Phys. Rev., 71. No 3, 1 Feb. 1947.

[4] They also found that negative muons do not get captured in Carbon and they thought that this contradicts Tomonaga's and Araki's prediction. However as it will become apparent this is due to the lower Z of the Carbon nucleus (Z=6) relative to Iron (Z=26).

It is given that $$a_0 = \frac{\hbar^2}{M_R e^2} \qquad (5)$$

Note this is the Bohr radius for a hydrogen atom where the electron has been replaced by a muon. The fact that the actuall Bohr radius for an atom with **Z** protons is α_0 / Z has already been taken in to account in the wavefunction formulae.

and (4) and (5) give: $$f = \frac{4}{3}\frac{Z^3 M_R^3 e^6}{\hbar^6} \times r_N^3 \qquad (6)$$

What is left now is to compute the nuclear radius r_N.

The nuclear radius can be estimated from the compton wavelength of the pion $(140\, MeV \Rightarrow r_p = 1.4\, fm)$ scaled by $A^{1/3}$ to account for the fact that the nucleus has many protons and neutrons:

$$\frac{4}{3}\pi r_N^3 = \frac{4}{3}\pi\left(\frac{\hbar}{m_\pi c} A^{\frac{1}{3}}\right)^3 \Rightarrow r_N^3 = \left(\frac{\hbar}{m_\pi c}\right)^3 A \qquad (7)$$

So from (6), (7) we get:

$$f = \frac{4}{3}\left(\frac{M_R}{m_\pi}\right)^3 A\left(\frac{Z}{137}\right)^3$$

The reduced mass is given by : $M_R = \dfrac{A m_p m_\mu}{A m_p + m_\mu}$ which for large **A** as in (b) becomes

$$M_R \approx m_\mu = 106\, MeV$$

Hence, $$f = \frac{4}{3}\left(\frac{106}{140}\right)^3 A\left(\frac{Z}{137}\right)^3 = \frac{4}{3} \times 0.43 \times A\left(\frac{Z}{137}\right)^3 = 0.58\, A\left(\frac{Z}{137}\right)^3$$

Easier way of solving this problem and comparison with the result from Perkins:

A more rough results can be computed as follows: The Bohr radius for such an atom is

$$\alpha_0 = 137 / Z\, m_\mu$$

and the nuclear radius can be computed as :

$$r_N = \frac{197.3\, MeV\ fm}{140\, MeV} A^{1/3} = 1.4\, fm\, A^{1/3}$$

Simple volume arguments then give that:

$$f = \left(\frac{r_N}{\alpha_0}\right)^3 = \left(1.4\left(\frac{1}{197.3\, MeV}\right)106\, MeV\right)^3 A\left(\frac{Z}{137}\right)^3 = 0.43\, A\left(\frac{Z}{137}\right)^3$$

which is off by (4/3) from the exact result.

It turns out that Perkins has used $1.2\, fm\, A^{1/3}$ for the nuclear radius which then gives: $f = 0.27\, A\left(\frac{Z}{137}\right)^3$. However, all these tell us that simple '*back of the envelope*' calculations can come very close to the exact quantum mechanical result (if you know what you are doing).

Next we will calculate the distance traveled by the muon in nuclear matter for a given time period t. Just like in Homework 1 we can compute the velocity of the muons as:

$$\frac{m_e v^2}{\alpha_0} = \frac{1}{4\pi\varepsilon_0} \times \frac{Z e^2}{\alpha_0^2} \quad \Rightarrow$$

$$m_e v^2 = \frac{1}{4\pi\varepsilon_0} \times \frac{Ze^2}{\alpha_0} \qquad (1)$$

As before structure constant is given by: $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137}$ which for

$\hbar = c = \varepsilon_0 = 1$ becomes

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137} \qquad (2)$$

However the Bohr radius (compute it and convince yourselves) also acquires a factor of Z in the denominator as seen before:

$$(1)(2) \Rightarrow m_e v^2 = \frac{Z\alpha}{\alpha_0} \Rightarrow v^2 = \frac{Z\alpha}{\alpha_0 \times m_e} = \frac{Z\alpha}{\frac{1}{\alpha \times Z \times m_e} \times m_e} = (Z\alpha)^2$$

Hence, $v = Z\alpha$ and the time that the particle spent in nuclear matter is of course

$$D = fct\left(\frac{Z}{137}\right)$$

(b) This is somewhat easier: The total width for the muon to disappear is the sum of the width to be eaten (captured) by the nucleus plus the width to die in old age by itself (decay). In other words:

$$\Gamma_{TOT} = \Gamma_{decay} + \Gamma_{capture} \Rightarrow$$

$$\frac{1}{\tau_{TOT}} = \frac{1}{\tau_{decay}} + \frac{1}{\tau_{capture}} \text{ with } \tau_{decay} = 2.16\,\mu\,sec \text{ and } \tau_{TOT} = 0.88\,\mu\,sec$$

Hence, $\tau_{capture} = 1.49\,\mu\,sec$ which is order of magnitude larger than a typical strong interaction lifetime. The mean free path can now be computed from:

$$f = 0.58\,A\left(\frac{Z}{137}\right)^3$$

$$D = fct\left(\frac{Z}{137}\right) = 0.58\,A\left(\frac{Z}{137}\right)^4 c\,\tau_{capture} = 0.58 \times 27 \times 3\,10^{10} \times (13/137)^4 \times 1.49\,\mu\,sec \Rightarrow$$

$$D \approx 57\,cm$$

Recall that the mean free path was inversely proportional to the interaction cross section. So the ratio of the cross section of the interaction that causes the capture over the strong interaction cross section is related to the mean free paths of the two interactions as

$$\frac{\sigma_{capture}}{\sigma_{strong}} = \frac{1\,fm}{57\,cm} = \left(\frac{1}{57}\right) \times 10^{-13}$$

As you may remember from quantum mechanics the cross section is proportional to the coupling-constant-square because the cross section is computed from the amplitude-square. Hence, the ratio of coupling of the two interactions is equal to the square root of the ratio of cross sections. Hence,

$$\frac{g_{capture}}{g_{strong}} = \sqrt{\frac{1}{57}\times 10^{-13}} = 4\times 10^{-8}$$

Conclusion: Whatever causes the muon to be captured in the nucleus and decay faster has nothing to do with the strong interaction because it has a coupling constant which is eight orders of magnitude smaller. So the muon could not have been the Yukawa particle. Actually the negative muon does decay faster than in vacuum when captured in the K-cell due to the reaction $\mu^- p \to n\, \nu_\mu$ which occurs in addition to the decay in vacuum which is described by $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. However, the former is also a weak interaction reaction and has a cross section far smaller than then strong interaction cross section. Hence, the muon is not the Yukawa particle.