Particle Physics Homework Assignment 2

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Problem 1: Show that $g_{\mu\nu} g^{\mu\nu} = 4$.

Solution:

$$g_{\mu\nu} g^{\mu\nu} = g_{\mu\nu} g^{\nu\mu} = \delta_{\mu}^{\mu} = 4$$

Problem 2: Show explicitly that $\Lambda^{\mu}{}_{\alpha} \Lambda_{\mu}{}^{\beta} = \delta_{\alpha}^{\beta}$. Use a Lorentz boost in the x-direction ($\vec{\beta} = \frac{v}{c}\hat{x}_0$) in the place of $\Lambda^{\mu}{}_{\nu}$.

Solution:

The problem in $\Lambda^{\mu}{}_{\alpha} \Lambda_{\mu}{}^{\beta} = \delta_{\alpha}^{\beta}$ is that this is not in a matrix multiplication form. In a matrix multiplication we sum over column-row indices (second-first index) where here the summation is over row-row indices. So just writing down two $\Lambda^{\mu}{}_{\alpha}$ matrices as given in the class and multiplying them is definitely wrong.

This problem can be solved in two ways:

1st way (short):

We have shown that $\Lambda^{\mu}{}_{\alpha} = \left(\Lambda^{-1}\right)_{\alpha}{}^{\mu}$ where $\left(\Lambda^{-1}\right)_{\alpha}{}^{\mu}$ is the inverse Lorentz matrix of $\Lambda^{\mu}{}_{\alpha}$. Therefore what we are really asked to prove is:

$$\Lambda^{\mu}{}_{\alpha} \Lambda_{\mu}{}^{\beta} = \delta_{\alpha}^{\beta} \Rightarrow \left(\Lambda^{-1}\right)_{\alpha}{}^{\mu} \Lambda_{\mu}{}^{\beta} = \delta_{\alpha}^{\beta} \qquad \text{(A)}$$

Now it is in a matrix multiplication form because we sum over the column elements of the first matrix multiplied one-by-one by the row elements of the second matrix. Relation (A) is already a proof but we will continue to prove it explicitly as requested.

Computing the inverse matrix can be easy using he following argument:

If the boost matrix for $\vec{\beta} = \frac{v}{c}\hat{x}_0$ is given by : $\Lambda^{\mu}{}_{\alpha} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (1)

then the inverse matrix must be given by the transformation defined by the boost:

$$\vec{\beta} = (-1)\times\frac{v}{c}\hat{x}_0 \qquad (2)$$

using (1) (2) we can derive the inverse matrix as:

$$(\Lambda^{-1})^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (3)$$

However what is needed is $(\Lambda^{-1})_{\alpha}^{\ \beta}$ and this means that we need to manipulate (3).

$$(\Lambda^{-1})_{\alpha}^{\ \beta} = g_{\alpha\mu}(\Lambda^{-1})^{\mu}_{\ \nu}g^{\nu\beta} \Rightarrow$$

$$(\Lambda^{-1})_{\alpha}^{\ \beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(\Lambda^{-1})_{\alpha}^{\ \beta} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4)$$

Now lets got back to (A). The fist matrix is exactly what we have in (4). However, the second is not exactly what we have in (1) and needs to be changed using the metric tensor as follows:

$$(\Lambda^{-1})_{\alpha}^{\ \mu}\Lambda_{\mu}^{\ \beta} = \delta_{\alpha}^{\ \beta} \quad \Rightarrow \quad (\Lambda^{-1})_{\alpha}^{\ \mu}\ g_{\mu\rho}\ \Lambda^{\rho}_{\ \sigma}\ g^{\beta\sigma} = \delta_{\alpha}^{\ \beta} \qquad (5)$$

The left-hand side of (5) can then be calculated as: $\left(\Lambda^{-1}\right)_{\alpha}^{\ \mu}\ g_{\mu\rho}\ \Lambda^{\rho}_{\ \sigma}\ g^{\beta\sigma} =$

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma^2-(\beta\gamma)^2 & +\beta\gamma^2-\beta\gamma^2 & 0 & 0 \\ -\beta\gamma^2+\beta\gamma^2 & \gamma^2-(\beta\gamma)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta_{\alpha}^{\ \beta}$$

Hence, we have the proof.

<u>Second way</u>: One can calculate all elements one-by-one and show that the result will be a $\delta_{\alpha}^{\ \beta}$. Here is how you would do it. Use:

$$\Lambda^{\mu}_{\ \alpha} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1)$$

to compute:

$$\Lambda_{\mu}^{\ \beta} = g_{\mu\rho}\Lambda^{\rho}_{\ \sigma}g^{\sigma\beta} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

Using (1) and (2) one can compute each element separately:

For example: $\Lambda^{\mu}_{\ 0}\Lambda_{\mu}^{\ 0} = \Lambda^{0}_{\ 0}\Lambda_{0}^{\ 0} + \Lambda^{1}_{\ 0}\Lambda_{1}^{\ 0} + \Lambda^{2}_{\ 0}\Lambda_{2}^{\ 0} + \Lambda^{3}_{\ 0}\Lambda_{3}^{\ 0} \Rightarrow$

$$\Lambda^{\mu}_{\ 0}\Lambda_{\mu}^{\ 0} = \gamma\gamma + (-\gamma\beta)(\beta\gamma) + 0 + 0 = \gamma^2-(\beta\gamma)^2 = 1 = \delta_{0}^{\ 0}$$

Hence we have proved that it is true for one matrix element and one can do this for all other elements.

Problem 3: Show that the expression $T^{\alpha\beta} x_\alpha y_\beta$ is a Lorentz invariant provided that $T^{\alpha\beta}$ transforms as a rank-2 tensor and x_α, y_β transform as covariant vectors.

Solution:

Let $T^{\alpha\beta}{}' x_\alpha{}' y_\beta{}'$ be the product in the reference frame **O'** and $T^{\alpha\beta} x_\alpha y_\beta$ the he same product in reference frame **O.** We are asked to show that this produce is Lorentz invariant that is: it does not change from frame to frame

$$T^{\alpha\beta}{}' x_\alpha{}' y_\beta{}' = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu T^{\mu\nu} \Lambda_\alpha{}^\sigma x_\sigma \Lambda_\beta{}^\rho x_\rho \qquad (1)$$

using the Lorentz property that $\Lambda^\mu{}_\alpha \Lambda_\mu{}^\beta = \delta_\alpha^\beta$ (2)

Using (1) (2) we have that:

$$T^{\alpha\beta}{}' x_\alpha{}' y_\beta{}' = \Lambda^\alpha{}_\mu \Lambda_\alpha{}^\sigma \Lambda^\beta{}_\nu \Lambda_\beta{}^\rho T^{\mu\nu} x_\sigma x_\rho = \delta_\mu^\sigma \delta_\nu^\rho T^{\mu\nu} x_\sigma x_\rho = T^{\mu\nu} x_\mu x_\nu$$

Hence, invariant.

Problem 4: Show that the 4-derivatives $\dfrac{\partial}{\partial x^\mu}$ and $\dfrac{\partial}{\partial x_\mu}$ transform as Lorentz covariant and contravariant vectors respectively.

Solution:

Consider the infinitesimal Lorentz transformation and its inverse:

$$\delta x^\mu{}' = \Lambda^\mu{}_\nu \delta x^\nu \Rightarrow \frac{\partial x^\mu{}'}{\partial x^\nu} = \Lambda^\mu{}_\nu \qquad (1)$$

$$\delta x^\mu = (\Lambda^{-1})^\mu{}_\nu \delta x^\nu{}' \Rightarrow \frac{\partial x^\mu}{\partial x^\nu{}'} = (\Lambda^{-1})^\mu{}_\nu \qquad (2)$$

<u>Thev ∂_ν transforms as a covariant vector because:</u>

$$\partial_\nu{}' = \frac{\partial}{\partial x^\nu{}'} = \frac{\partial x^\mu}{\partial x^\nu{}'}\frac{\partial}{\partial x^\mu} = (\Lambda^{-1})^\mu{}_\nu \frac{\partial}{\partial x^\mu} = (\Lambda^{-1})^\mu{}_\nu \partial_\mu = \Lambda_\nu{}^\mu \partial_\mu$$

Similarly one can compute the inverse transformation (not required):

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \frac{\partial x^{\nu\prime}}{\partial x_{\mu}}\frac{\partial}{\partial x^{\nu\prime}} = \Lambda^{\nu\mu}\frac{\partial}{\partial x^{\nu\prime}} = \Lambda^{\nu\mu}\partial_{\nu}{}' = (\Lambda^{-1})^{\mu\nu}\partial_{\nu}{}'$$

<u>The contravariant derivate transformation can be computed from</u>:

$$\partial^{\nu\prime} = \frac{\partial}{\partial x_{\nu}{}'} = \frac{\partial x_{\mu}}{\partial x_{\nu}{}'}\frac{\partial}{\partial x_{\mu}} = (\Lambda^{-1})_{\mu}{}^{\nu}\frac{\partial}{\partial x_{\mu}} = (\Lambda^{-1})_{\mu}{}^{\nu}\partial^{\mu} = \Lambda^{\nu}{}_{\mu}\partial^{\mu}$$

in other words we have proven that it transforms as a contravariant vector.

Problem 5:

1) Write down the definition of a parity transformation.
2) Consider two Lorentz 4-vectors: X^{μ} and Y^{μ}. X^{μ} transforms as a polar vector, and Y^{μ} as an axial vector . How do they transform under parity ?
3) Which of the following Lorentz invariant quantities is invariant under parity and which is not:

$$(a)\ X^{\mu}X_{\mu}\ \ (b)\ Y^{\mu}Y_{\mu}\ \ (c)\ (X^{\mu}-Y^{\mu})\cdot(X_{\mu}-Y_{\mu})$$

Solution:

1) A parity transformation is defined as:

$$P:\ \ t \rightarrow t' = t \quad \vec{x} \rightarrow \vec{x}' = -\vec{x}$$

2) The polar vector transforms under parity as:

$$P(X^{\mu}) = P(X^0;\vec{X}) = (X^0;-\vec{X})$$

whilst the axial vector transforms as:

$$P(Y^{\mu}) = P(Y^0;\vec{Y}) = (-Y^0;+\vec{Y})$$

3) Next consider the quantities:

(a) $P(X_{\mu}X^{\mu}) = (X^0X^0-(-1)\vec{X}\cdot(-1)\vec{X}) = X_{\mu}X^{\mu}$ (invariant under parity)

(b) $\boldsymbol{P}(Y_\mu Y^\mu) = ((-1)Y^0(-1)Y^0 - \vec{Y}\cdot\vec{Y}) = Y_\mu Y^\mu$ (invariant under parity)

(c) $\boldsymbol{P}(X_\mu Y^\mu) = ((+1)X^0(-1)Y^0 - (-1)\vec{X}\cdot(+1)\vec{Y}) = -X_\mu Y^\mu$

(not invariant under parity)

Hence,

$$\boldsymbol{P}[(X^\mu - Y^\mu)\cdot(X_\mu - Y_\mu)] = \boldsymbol{P}[X_\mu X^\mu + Y_\mu Y^\mu - X_\mu Y^\mu - Y_\mu X^\mu] \Rightarrow$$

$$\boldsymbol{P}[(X^\mu - Y^\mu)\cdot(X_\mu - Y_\mu)] = X_\mu X^\mu + Y_\mu Y^\mu + 2\ X_\mu Y^\mu = [(X^\mu + Y^\mu)\cdot(X_\mu + Y_\mu)]$$

(obviously not invariant under parity)

Problem 6:

1) Using Maxwell's equation in three dimensions show that the Electric Field, $\vec{E}$, is a vector and the magnetic field, $\vec{B}$, an axial vector.
2) As one can see, Maxwell's equations are not completely symmetric because although they include an electric charge density, ρ_e, and an electric current density $\vec{J}_e$, the equivalent magnetic quantities, ρ_m, $\vec{J}_m$, are absent indicating that there are no magnetic monopols. Introduce magnetic monopols and write down the completely symmetric Maxwell equations. Show that ρ_m must be a pseudoscalar and $\vec{J}_m$ an axial vector.

Solution:

Maxwell's equations in the standard 3-D notation are:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \qquad (1)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (2)$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad (3)$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad (4)$$

The electric charge density, $\rho_e(\vec{x})$, is a scalar whilst the electric current density, $\vec{J}_e(\vec{x})$, is a polar vector.

By requiring that equation (1) is invariant under rotations and parity (electromagnetic interactions conserve parity) we have that the electric field, $\vec{E}(\vec{x})$, is a polar vector. If the electric field is a polar vector, parity invariance of (3) implies that the magnetic field, $\vec{B}(\vec{x})$, is an axial vector. At the end one can easily check that equations (2) and (4) are invariant under parity provided that $\vec{E}(\vec{x})$ is a polar vector and $\vec{B}(\vec{x})$ is an axial vector. Interstingly enough it is enough to take the electric charge density, $\rho_e(\vec{x})$ as a scalar, which it is by definition and the rest follow.

Next we introduce the magnetic charge density and the magnetic current density:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e \qquad (1)$$

$$\vec{\nabla}\cdot\vec{B} = 4\pi\rho_m \qquad (2)$$

$$\vec{\nabla}\times\vec{E} = \frac{4\pi}{c}\vec{J}_m - \frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad (3)$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_e + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad (4)$$

In this case (2) implies that ρ_m must be a pseudo-scalar if the magnetic filed is an axial vector and (2) is invariant under parity. Similarly if (3) is invariant under parity and the electric field is a polar vector, it must be that the magnetic current density, $\vec{J}_m$, is an axial vector.