Particle Physics Homework Assignment 11

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Problem 1: In homework assignment 10 we have shown that the CP transformation of a negative helicity and massless neutrino results to positive helicity antineutrino which is described by

$$\Psi_{CP}(x) = -v^{(2)}(-\vec{p}\,;m=0)\,e^{+ip^0x^0-i(-\vec{p})\cdot\vec{x}}$$

Apply a Time Reversal transformation on $\Psi_{CP}(x)$ to derive the TCP transformed spinor $\Psi_{TCP}(x)$.

Solution:

$$\Psi_{CP}(x) = -v^{(2)}(-\vec{p}\,;m=0)\,e^{+i(p^0x^0-(-\vec{p})\cdot\vec{x})} \tag{1}$$

$$T = i\gamma^1\gamma^3 = i\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \tag{2}$$

From (1) and (2) we have that

$$\Psi_{TCP}(x) = T\,\Psi^*(\vec{x},-t) = -\begin{pmatrix} -\sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}\sqrt{E}\begin{pmatrix} -\vec{\sigma}^*\cdot\hat{p} \\ 1 \end{pmatrix}\chi^1 e^{-i(p^0(-x^0)-(-\vec{p})\cdot\vec{x})} \Rightarrow$$

$$\Psi_{TCP}(x) = -\sqrt{E}\begin{pmatrix} \sigma^2\vec{\sigma}^*\cdot\hat{p} \\ -\sigma^2 \end{pmatrix}\chi^1 e^{-i(p^0(-x^0)-(-\vec{p})\cdot\vec{x})} \Rightarrow$$

$$\Psi_{TCP}(x) = -\sqrt{E}\begin{pmatrix} \vec{\sigma}\cdot\hat{p}\,(-\sigma^2) \\ -\sigma^2 \end{pmatrix}\chi^1 e^{-i(p^0(-x^0)-(-\vec{p})\cdot\vec{x})} \Rightarrow$$

$$\Psi_{TCP}(x) = -\sqrt{E}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} \\ 1 \end{pmatrix}(-\sigma^2\chi^1)\,e^{-i(p^0(-x^0)-(-\vec{p})\cdot\vec{x})} \Rightarrow$$

$$\Psi_{TCP}(x) = i\sqrt{E}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} \\ 1 \end{pmatrix}\chi^2 e^{-i(p^0(-x^0)-(-\vec{p})\cdot\vec{x})} \Rightarrow$$

$$\Psi_{TCP}(x) = i\,u^{(4)}(-\vec{p})\,e^{-i\left(p^0(-x^0)-(-\vec{p})\cdot\vec{x}\right)} = i\,v^{(1)}(\vec{p})\,e^{i\,p\,x}$$

Figure 1 shows how a negative helicity neutrino will change under Parity, Charge Conjugation and Time Reversal.

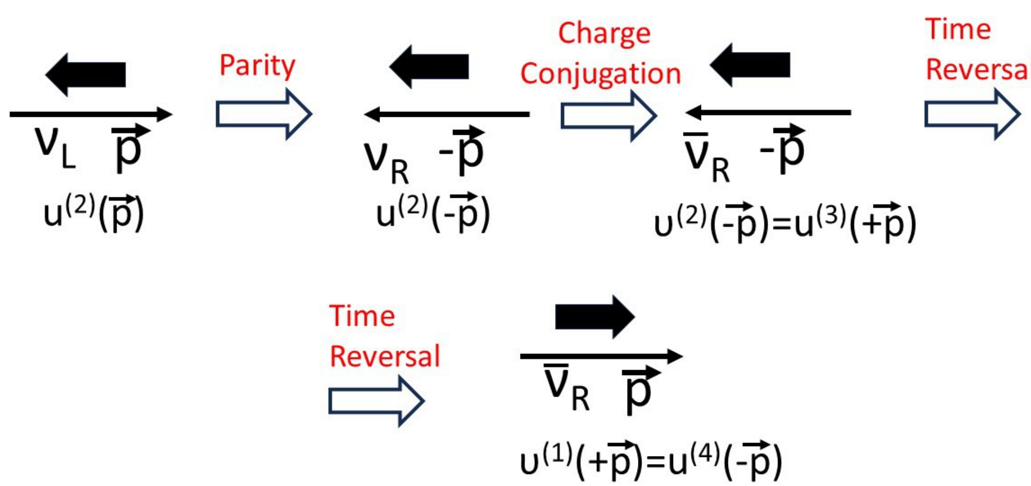

Figure 1: *The negative helicity neutrino spinor under Parity, Charge Conjugation and Time Reverasal. Thick arrows indicate the spin direction.*

Problem 2: Consider a negative energy electron coupled to an electromagnetic field. The electron is described by the Dirac equation

$$\left[\vec{a}\cdot(-i\vec{\nabla}-e\vec{A}(x))+\beta m+e\Phi(x)\right]\Psi(x) = -E\,\Psi(x) \quad \text{where} \quad E>0$$

1. Show, by requiring that this equation is invariant under TCP, that electromagnetic field transforms under CPT as $A^{\mu}_{TCP}(-x) = -A^{\mu}(x)$.

2. The TPC transformed electron corresponds to a positive energy solution.

Solution:

So we start from

$$\left[\vec{a}\cdot(-i\vec{\nabla}-e\vec{A}(x))+\beta m+e\Phi(x)\right]\Psi(x) = -E\,\Psi(x) \tag{1}$$

and we will substitute $\Psi(x)$ with

$$\Psi_{CPT}(x') = i\gamma^5\Psi(x) \Rightarrow \Psi(x) = -i\gamma^5\Psi_{CPT}(x') \quad \text{where} \quad x' = -x \tag{2}$$

From (1) and (2) we have that

$$\left[\vec{a}\cdot(-i\vec{\nabla}-e\vec{A}(x))+\beta m+e\Phi(x)\right](-i)\gamma^5\Psi_{CPT}(x') = -E(-i)\gamma^5\Psi_{CPT}(x') \quad \Rightarrow$$

$$(-i)\gamma^5\left[\gamma^0\vec{\gamma}\cdot(-i\vec{\nabla}-e\vec{A}(x))-\gamma^0 m+e\Phi(x)\right]\Psi_{CPT}(x') = -E(-i)\gamma^5\Psi_{CPT}(x') \quad \Rightarrow$$

$$\left[\gamma^0\vec{\gamma}\cdot(-i\vec{\nabla}-e\vec{A}(x))-\gamma^0 m+e\Phi(x)\right]\Psi_{CPT}(x') = -E\,\Psi_{CPT}(x') \quad \Rightarrow$$

$$\left[\gamma^0\vec{\gamma}\cdot(+i\vec{\nabla}'-e\vec{A}(-x'))-\gamma^0 m+e\Phi(-x')\right]\Psi_{CPT}(x') = -E\,\Psi_{CPT}(x') \quad \Rightarrow$$

$$\left[\gamma^0\vec{\gamma}\cdot(-i\vec{\nabla}'+e\vec{A}(-x'))+\gamma^0 m-e\Phi(-x')\right]\Psi_{CPT}(x') = E\,\Psi_{CPT}(x') \tag{3}$$

By comparing (3) with one we conclude that if the equation is to be CPT invariant it must be that

$$\vec{A}_{CPT}(x') = -\vec{A}(-x') \quad \text{and} \quad \Phi_{CPT}(x') = -\Phi(-x')$$

while the spinor transforms as shown in (2).

Problem 3: Show that

a) $$F^{\mu\nu}\tilde{F}_{\mu\nu} = \vec{E}\cdot\vec{B}$$

b) this term violates both Parity and Time Reversal symmetries.

$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ is the Maxwell tensor and $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ its dual. $\vec{E}\,,\vec{B}$ are the electric and magnetic fields respectively.

Solution: It is easy to show that

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

and that

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

therefore

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \begin{pmatrix} \vec{E}\cdot\vec{B} & 0 & 0 & 0 \\ 0 & \vec{E}\cdot\vec{B} & 0 & 0 \\ 0 & 0 & \vec{E}\cdot\vec{B} & 0 \\ 0 & 0 & 0 & \vec{E}\cdot\vec{B} \end{pmatrix} = \vec{E}\cdot\vec{B}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Clearly this violates Parity because the electric field is a polar vector whilst the magnetic filed is an axial vector. It also violates Time Reversal because under Time Reversal the electric filed does not change whilst the magnetic filed reverses its sign. Therefore using the CPT theorem we conclude that it also violates CP.