Particle Physics Homework Assignment 10

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Problem 1: Show that $\sigma^2 \vec{\sigma}^* = -\vec{\sigma}\sigma^2$

Solution:

The Pauli matrices satisfy

$$(\sigma^1)^* = \sigma^1 \ , \ (\sigma^2)^* = -\sigma^2 \ , \ (\sigma^3)^* = \sigma^3 \quad \text{and} \quad \sigma^i \sigma^j = i\,\varepsilon^{ijk}\sigma^k \ .$$

Therefore we have

(a) $\sigma^2(\sigma^2)^* = -\sigma^2\sigma^2$

(b) $\sigma^2(\sigma^1)^* = \sigma^2\sigma^1 = i\,\varepsilon^{213}\sigma^3 = -i\,\varepsilon^{123}\sigma^3 = -i\,\sigma^3$ (1)

$\sigma^1\sigma^2 = i\,\varepsilon^{123}\sigma^3 = i\,\sigma^3$ (2)

(1) (2) $\Rightarrow \sigma^2(\sigma^1)^* = -\sigma^1\sigma^2$

(c) $\sigma^2(\sigma^3)^* = \sigma^2\sigma^3 = i\,\varepsilon^{231}\sigma^1 = i\,\varepsilon^{123}\sigma^1 = i\,\sigma^1$ (3)

$\sigma^3\sigma^2 = i\,\varepsilon^{321}\sigma^1 = -i\,\sigma^1$ (4)

(3) (4) $\Rightarrow \sigma^2(\sigma^3)^* = -\sigma^3\sigma^2$

Problem 2: Show that in the Pauli Dirac representation the matrix C satisfies

$$C = -C^{-1} = -C^{+} = -C^{T}$$

Solution:

a) $CC^{-1} = 1 \;\Rightarrow\; i\gamma^2\gamma^0 C^{-1} = 1 \;\Rightarrow\; C^{-1} = (-i)(-1)(-1)\gamma^2\gamma^0 = -C$

b) $C^{+} = (i\gamma^2\gamma^0)^{+} = -i(\gamma^0)^{+}(\gamma^2)^{+} = -i\gamma^0\gamma^0\gamma^2\gamma^0 = -i\gamma^2\gamma^0 = -C$

c) $C^{T} = (i\gamma^2\gamma^0)^{T} = i(\gamma^0)^{T}(\gamma^2)^{T} = i\gamma^0\gamma^2 = -i\gamma^2\gamma^0 = -C$

Problem 3: Show that

$\Psi_c = C\bar{\Psi}^T$ and $\bar{\Psi}_c = -\Psi^T C^{-1}$

Solution:

a) $\Psi_c = C\gamma^0\Psi^{*} = C(\gamma^0)^T(\Psi^{+})^T = C(\Psi^{+}\gamma^0)^T = C\bar{\Psi}^T$

b) $\bar{\Psi}_c = (C\gamma^0\Psi^{*})^{+}\gamma^0 = \Psi^T(\gamma^0)^{+}C^{+} = -\Psi^T\gamma^0 C\gamma^0 = \Psi^T\gamma^0\gamma^0 C = -\Psi^T C^{-1}$

Problem 4: As shown in Homework Assignment 9 the spinor

$$\boldsymbol{\Psi}(x) = \sqrt{E}\begin{pmatrix} 1 \\ \vec{\sigma}\cdot\hat{p} \end{pmatrix}\chi^2 e^{-ipx}$$

has negative helicity and can describe a neutrino with negative helicity which has been detected in nature. Show that the charge conjugate of this spinor represents an anti-neutrino with negative helicity which has not been detected in nature. This means that the interaction which is responsible for the production of neutrinos violates the charge conjugation symmetry.

Solution:

$$\boldsymbol{\Psi}_C(x) = [\boldsymbol{\Psi}(x)]^c = i\gamma^2[\boldsymbol{\Psi}(x)]^* = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\sqrt{E}\begin{pmatrix} 1 \\ \vec{\sigma}^*\cdot\hat{p} \end{pmatrix}\chi^2 e^{+ipx} \Rightarrow$$

$$\boldsymbol{\Psi}_C(x) = i\sqrt{E}\begin{pmatrix} \sigma^2\vec{\sigma}^*\cdot\hat{p} \\ -\sigma^2 \end{pmatrix}\chi^2 e^{+ipx} = i\sqrt{E}\begin{pmatrix} \vec{\sigma}\cdot\hat{p}\,(-\sigma^2) \\ -\sigma^2 \end{pmatrix}\chi^2 e^{+ipx} \Rightarrow$$

$$\boldsymbol{\Psi}_C(x) = \sqrt{E}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} \\ 1 \end{pmatrix}(-i\sigma^2)\chi^2 e^{+ipx} = -\sqrt{E}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} \\ 1 \end{pmatrix}\chi^1 e^{+ipx} \Rightarrow$$

$$\boldsymbol{\Psi}_C(x) = -u^3(-\vec{p}\,;m=0)\,e^{+ipx} = -v^2(\vec{p}\,;m=0)\,e^{+ipx}$$

The resulting spinor has positive energy and negative helicity. This can be demonstrated as follows.

$$\vec{\Sigma}\cdot\hat{p}\;\; u^3(\vec{p}\,;m=0) = u^3(\vec{p}\,;m=0) \Rightarrow$$

$$-\vec{\Sigma}\cdot\hat{p}\;\; u^3(-\vec{p}\,;m=0) = u^3(-\vec{p}\,;m=0) \Rightarrow$$

$$\vec{\Sigma}\cdot\hat{p}\;\; v^2(\vec{p}\,;m=0) = -v^2(\vec{p}\,;m=0)$$

Hence, the spinor we obtain after charge conjugating a negative helicity neutrino spinor is an anti-neutrino spinor with negative helicity. Experimentally we have not observed anti-neutrinos with negative helicity so the weak interaction violates change conjugation symmetry. In this discussion we have assumed that the mass of the neutrino is zero. Today we know that neutrinos have non-zero mass however small it may be. Hence, in principle, although suppressed as shown in Lecture 9, anti-neutrinos with negative helicity should exist in nature. This example demonstrates also that charge conjugation changes particle to anti-particle without altering helicity.

Problem 5: Use the charge conjugate spinor of a neutrino with negative helicity from the previous problem

$$\Psi_C(x) = -\sqrt{E}\begin{pmatrix}\vec{\sigma}\cdot\hat{p}\\ 1\end{pmatrix}\chi^1 e^{+ipx}$$

which as we have seen has negative helicity and calculate its parity inverted spinor Ψ_{PC} .

Solution:

$$\Psi_{PC}(x) = (-1)\gamma^0\left[\sqrt{E}\begin{pmatrix}\vec{\sigma}\cdot\hat{p}\\ 1\end{pmatrix}\chi^1 e^{+ipx'}\right]_{\vec{x}'=-\vec{x},\, x^{0\prime}=x^0} \Rightarrow$$

$$\Psi_{PC}(x) = (-1)\sqrt{E}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\vec{\sigma}\cdot\hat{p}\\ 1\end{pmatrix}\chi^1\left[e^{+ipx'}\right]_{\vec{x}'=-\vec{x},\, x^{0\prime}=x^0} \Rightarrow$$

$$\Psi_{PC}(x) = (-1)\sqrt{E}\begin{pmatrix}\vec{\sigma}\cdot\hat{p}\\ -1\end{pmatrix}\chi^1 e^{+ip^0x^0-i\vec{p}\cdot(-\vec{x})} \Rightarrow$$

$$\Psi_{PC}(x) = \sqrt{E}\begin{pmatrix}\vec{\sigma}\cdot(-\hat{p})\\ 1\end{pmatrix}\chi^1 e^{+ip^0x^0-i(-\vec{p})\cdot\vec{x}} \Rightarrow$$

$$\Psi_{PC}(x) = u^{(3)}(\vec{p}\,; m=0)\, e^{+ip^0x^0-i(-\vec{p})\cdot\vec{x}} \Rightarrow$$

$$\Psi_{PC}(x) = v^{(2)}(-\vec{p}\,; m=0)\, e^{+ip^0x^0-i(-\vec{p})\cdot\vec{x}}$$

However,

$$\vec{\Sigma}\cdot\hat{p}\;\; u^3(\vec{p}\,; m=0) = u^3(\vec{p}\,; m=0) \;\Rightarrow\; \vec{\Sigma}\cdot\hat{p}\;\; v^2(-\vec{p}\,; m=0) = v^2(-\vec{p}\,; m=0)$$

Hence, the combined PC operation transformed a negative helicity neutrino to a positive helicity anti-neutrino which exists in nature. Alternatively, one could have started by applying parity first and then charge conjugation. The result would have been the same up to a phase.

$$\Psi_{CP}(x) = -v^{(2)}(-\vec{p}\,; m=0)\, e^{+ip^0x^0-i(-\vec{p})\cdot\vec{x}}$$

Problem 6: Consider the Majorana representation of the Dirac matrices which is given by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \ , \ \gamma^1 = i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \ , \ \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \ , \ \gamma^3 = -i\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}$$

Show that in this representation $\Psi_c = \Psi^*$. In this representation one can define a spinor $\chi = \Psi + \Psi_c$. Show that χ, provided that it represents a neutral particle, is also a solution of the Dirac equation which is real and satisfies $\chi = \chi_c$. In other words it represents a particle which is identical to its antiparticle.

Solution: In a similar was as in the lecture we setup the equations for particle and antiparticle.

$$[i\gamma^\mu(\partial_\mu - i e A_\mu) - m]\Psi(x) = 0 \qquad (1)$$

$$[i\gamma^\mu(\partial_\mu + i e A_\mu) - m]\Psi_c(x) = 0 \qquad (2)$$

$$\Psi_c = C\gamma^0\Psi^* \Rightarrow \Psi^* = \gamma^0 C^{-1}\Psi_c \qquad (3)$$

From (1) and (3) we have that

$$[-i(\gamma^\mu)^*(\partial_\mu + i e A_\mu) - m]\gamma^0 C^{-1}\Psi_c(x) = 0 \qquad (4)$$

In the Majorana representation $(\gamma^0)^T = -\gamma^0$.

Therefore $(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0 \Rightarrow (\gamma^\mu)^* = \gamma^0(\gamma^\mu)^T\gamma^0 \qquad (5)$

From (4) and (5) we have that

$$[-i\gamma^0(\gamma^\mu)^T\gamma^0(\partial_\mu + i e A_\mu) - m]\gamma^0 C^{-1}\Psi_c(x) = 0 \Rightarrow$$

$$[-iC\gamma^0\gamma^0(\gamma^\mu)^T\gamma^0\gamma^0 C^{-1}(\partial_\mu + i e A_\mu) - m]\Psi_c(x) = 0 \Rightarrow$$

$$[-iC(\gamma^\mu)^T C^{-1}(\partial_\mu + i e A_\mu) - m]\Psi_c(x) = 0 \qquad (6)$$

By comparing (2) and (6) we conclude that if the transformation described in (3) exists if one can find a matrix C which satisfies

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \qquad (7)$$

This result is no surprise of course since we have shown already in Lecture 10 in a representation independent way that this must be true if a charge conjugation operator is to be found.

In the Majorana representation we have that

$$(\gamma^0)^T = -\gamma^0 \quad (\gamma^1)^T = +\gamma^1 \quad (\gamma^2)^T = +\gamma^2 \quad (\gamma^3)^T = +\gamma^3 \qquad (8)$$

Hence, from (7) and (8) we have that

$$[C,\gamma^0] = 0 \;,\; \{C,\gamma^1\} = 0 \;,\; \{C,\gamma^2\} = 0 \;,\; \{C,\gamma^3\} = 0 \qquad (9)$$

Clearly (9) implies that

$$C = \gamma^0 \qquad (10)$$

Substituting (10) in to (3) one gets

$$\Psi_c = C\gamma^0\Psi^* \Rightarrow \Psi_c = \Psi^*$$

Hence, in the Majorana representation the charge conjugation operation is simply the complex conjugation operation.

Next define

$$\chi = \Psi + \Psi_c = \Psi + \Psi^* \Rightarrow \chi = \chi_c$$

χ is clearly real and is a solution of the Dirac equation if χ is neutral. Hence, χ **represents a neutral fermion which is identical to its antiparticle.** People have speculated that the neutrino may be a majorana particle but this has not been confirmed experimentally.

Problem 7: Show that the particle and antiparticle spinors can be expressed as

(a) $$u^{(s)}(\vec{p},m) = \frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} \times u^{(s)}(0,m)$$

(b) $$v^{(s)}(\vec{p},m) = \frac{-\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} \times v^{(s)}(0,m)$$

where

$$u^{(1)}(0,m) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)}(0,m) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$v^{(1)}(0,m) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v^{(2)}(0,m) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Solution:

a) $$\frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} = \frac{1}{\sqrt{E+M}}\left(p^0\gamma^0-\vec{\gamma}\cdot\vec{p}+m\right) = \frac{1}{\sqrt{E+M}}\begin{bmatrix} p^0+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p^0+m \end{bmatrix} \Rightarrow$$

$$\frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} = \sqrt{E+m}\begin{bmatrix} 1 & \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \frac{-p^0+m}{E+m} \end{bmatrix} \Rightarrow$$

$$\sqrt{E+m}\begin{bmatrix} 1 & \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \frac{-p^0+m}{E+m} \end{bmatrix} \times u^{(s)}(0,m) = \sqrt{E+m}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \times \chi^s = u^{(s)}(\vec{p},m)$$

where $\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $$\frac{-\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} = \frac{1}{\sqrt{E+M}}\left(-p^{0}\gamma^{0}+\vec{\gamma}\cdot\vec{p}+m\right) = \frac{1}{\sqrt{E+M}}\begin{bmatrix} -p^{0}+m & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & p^{0}+m \end{bmatrix} \Rightarrow$$

$$\frac{-\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} = \sqrt{E+m}\begin{bmatrix} \dfrac{-p^{0}+m}{p^{0}+m} & \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \dfrac{-\vec{\sigma}\cdot\vec{p}}{E+m} & 1 \end{bmatrix} \Rightarrow$$

$$\frac{-\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} v^{(s)}(0,m) = \sqrt{E+m}\begin{bmatrix} \dfrac{-p^{0}+m}{p^{0}+m} & \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \dfrac{-\vec{\sigma}\cdot\vec{p}}{E+m} & 1 \end{bmatrix} v^{(s)}(0,m) \Rightarrow$$

$$\frac{-\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} v^{(s)}(0,m) = \sqrt{E+m}\begin{bmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{bmatrix} v^{(s)}(0,m) = v^{(s)}(\vec{p},m)$$

Problem 8: Show that

$$(\gamma^\mu p_\mu + m)\gamma^0(\gamma^\mu p_\mu + m) = 2E(\gamma^\mu p_\mu + m)$$

Solution:

$$(\gamma^\mu p_\mu + m)\gamma^0(\gamma^\mu p_\mu + m) = \gamma^\mu p_\mu \gamma^0 \gamma^\nu p_\nu + m(\gamma^\mu \gamma^0 + \gamma^0 \gamma^\mu) p_\mu + m^2 \gamma^0 \qquad (1)$$

$$\gamma^\mu p_\mu \gamma^0 \gamma^\nu p_\nu = \gamma^\mu \gamma^0 \gamma^\nu p_\mu p_\nu = \gamma^\mu(-\gamma^\nu \gamma^0 + 2g^{\nu 0}) p_\mu p_\nu \quad \Rightarrow$$

$$\gamma^\mu p_\mu \gamma^0 \gamma^\nu p_\nu = -\gamma^\mu \gamma^\nu \gamma^0 p_\mu p_\nu + 2p^0 \gamma^\mu p_\mu = -p^2 \gamma^0 + 2E\gamma^\mu p_\mu \quad \Rightarrow$$

$$\gamma^\mu p_\mu \gamma^0 \gamma^\nu p_\nu = -m^2 \gamma^0 + 2E\gamma^\mu p_\mu \qquad (2)$$

$$m(\gamma^\mu \gamma^0 + \gamma^0 \gamma^\mu) p_\mu = m\, 2g^{\mu 0} p_\mu = 2m\, p^0 = 2Em \qquad (3)$$

and from (1), (2), and (3) we have that

$$(\gamma^\mu p_\mu + m)\gamma^0(\gamma^\mu p_\mu + m) = 2Em + 2E\gamma^\mu p_\mu = 2E(\gamma^\mu p_\mu + m)$$

Problem 9: Show that

$$\text{(a)} \qquad \sum_{\alpha=1}^{2} u^{(\alpha)}(\vec{p},m) \otimes \bar{u}^{(\alpha)}(\vec{p},m) = \gamma^{\mu} p_{\mu} + m$$

$$\text{(b)} \qquad \sum_{\alpha=1}^{2} v^{(\alpha)}(\vec{p},m) \otimes \bar{v}^{(\alpha)}(\vec{p},m) = \gamma^{\mu} p_{\mu} - m$$

Solution:

(a) From exercise 7 we have that

$$u^{(s)}(\vec{p},m) = \frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} \times u^{(s)}(0,m) \Rightarrow \bar{u}^{(s)}(\vec{p},m) = \bar{u}^{(s)}(0,m) \times \frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} \qquad (1)$$

Hence, using (1) we have that

$$\sum_{\alpha=1}^{2} u^{(\alpha)}(\vec{p},m) \otimes \bar{u}^{(\alpha)}(\vec{p},m) = \frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} \times \sum_{\alpha=1}^{2} u^{(\alpha)}(0,m) \otimes \bar{u}^{(\alpha)}(0,m) \times \frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} \qquad (2)$$

However,

$$\sum_{\alpha=1}^{2} u^{(\alpha)}(0,m) \otimes \bar{u}^{(\alpha)}(0,m) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1+\gamma^{0}}{2} \qquad (3)$$

and from (2) and (3) we have that

$$\sum_{\alpha=1}^{2} u^{(\alpha)}(\vec{p},m) \otimes \bar{u}^{(\alpha)}(\vec{p},m) = \frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} \times \frac{1+\gamma^{0}}{2} \times \frac{\gamma^{\mu} p_{\mu}+m}{\sqrt{E+m}} =$$

$$\frac{1}{2(E+M)}[(\gamma^{\mu} p_{\mu}+m)(\gamma^{\mu} p_{\mu}+m)+(\gamma^{\mu} p_{\mu}+m)\gamma^{0}(\gamma^{\mu} p_{\mu}+m)] \qquad (4)$$

However,

$$(\gamma^{\mu} p_{\mu}+m)(\gamma^{\mu} p_{\mu}+m) = 2m(\gamma^{\mu} p_{\mu}+m) \qquad (5)$$

Using the results from exercise 8 and (4) and (5) we have that

$$\sum_{\alpha=1}^{2} u^{(\alpha)}(\vec{p},m)\otimes\bar{u}^{(\alpha)}(\vec{p},m) = \frac{1}{2(E+M)}[2m(\gamma^{\mu}p_{\mu}+m)+2E(\gamma^{\mu}p_{\mu}+m)] \Rightarrow$$

$$\sum_{\alpha=1}^{2} u^{(\alpha)}(\vec{p},m)\otimes\bar{u}^{(\alpha)}(\vec{p},m) = \gamma^{\mu}p_{\mu}+m$$

(b) From exercise (7) we have that

$$v^{(s)}(\vec{p},m) = \frac{-\gamma^{\mu}p_{\mu}+m}{\sqrt{E+m}} \times v^{(s)}(0,m) \quad , \quad \bar{v}^{(s)}(\vec{p},m) = \bar{v}^{(s)}(0,m) \times \frac{-\gamma^{\mu}p_{\mu}+m}{\sqrt{E+m}}$$

and using them we get

$$\sum_{\alpha=1}^{2} v^{(\alpha)}(\vec{p},m)\otimes\bar{v}^{(\alpha)}(\vec{p},m) =$$

$$\frac{(-\gamma^{\mu}p_{\mu}+m)}{\sqrt{E+m}}[\sum_{\alpha=1}^{2} v^{(\alpha)}(\vec{p},0)\otimes\bar{v}^{(\alpha)}(\vec{p},0)]\frac{(-\gamma^{\mu}p_{\mu}+m)}{\sqrt{E+m}} \tag{1}$$

However,

$$[\sum_{\alpha=1}^{2} v^{(\alpha)}(\vec{p},0)\otimes\bar{v}^{(\alpha)}(\vec{p},0)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \frac{\gamma^{0}-1}{2} \tag{2}$$

and by substituting (2) into (1) you get

$$\sum_{\alpha=1}^{2} v^{(\alpha)}(\vec{p},m)\otimes\bar{v}^{(\alpha)}(\vec{p},m) = \frac{1}{E+m}(-\gamma^{\mu}p_{\mu}+m)\frac{\gamma^{0}-1}{2}(-\gamma^{\mu}p_{\mu}+m) \tag{3}$$

It is easy to show that

$$(-\gamma^{\mu}p_{\mu}+m)\gamma^{0}(-\gamma^{\mu}p_{\mu}+m) = 2E(+\gamma^{\mu}p_{\mu}-m) \tag{4}$$

$$(-\gamma^{\mu}p_{\mu}+m)(-\gamma^{\mu}p_{\mu}+m) = 2m(m-\gamma^{\mu}p_{\mu}) \tag{5}$$

and from (3), (4) and (5) we have that

$$\sum_{\alpha=1}^{2} v^{(\alpha)}(\vec{p},m)\otimes\bar{v}^{(\alpha)}(\vec{p},m) = \frac{1}{2(E+m)}[2E(\gamma^{\mu}p_{\mu}-m)-2m(-\gamma^{\mu}p_{\mu}+m)] \Rightarrow$$

$$\sum_{\alpha=1}^{2} v^{(\alpha)}(\vec{p},m)\otimes\bar{v}^{(\alpha)}(\vec{p},m) = (\gamma^{\mu}p_{\mu}-m)$$