Particle Physics Homework Assignment 1

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Problem 1: (a) Show that $1\,Kgr = 5.6\,10^{26}\,GeV$

(b) In a unit system where $\hbar = c = 1$ show that:

I. $1\,Gev^{-2} = 0.389\,mb$

II. $1\,m = 5.068\,10^{15}\,GeV^{-1}$

III. $1\,sec = 1.5\,10^{24}\,GeV^{-1}$

Use $\hbar c = 197.3\,MeV\ fm$.

Solution:

(a) $$E \;=\; 1\text{Kgr}\left(3\,10^{8}\,\frac{m}{sec}\right)^{2} \;=\; 9\;\;10^{16}\,J \qquad (1)$$

and $$1\,eV \;=\; 1.6\,10^{-19}\,Cb\,Volt \;=\; 1.6\,10^{-19}\,J \qquad (2)$$

so $$(1)(2) \;\Rightarrow\; E(1\text{Kgr}) \;=\; \frac{9\,10^{16}}{1.6\,10^{-19}}\,eV \;=\; 5.6\,10^{35}\,eV \;=\; 5.6\,10^{26}\,GeV$$

(b)

I. $\hbar c = 197.3\,MeV\ fm$ (1)

$\hbar c = 1$ (2)

$$(1)(2) \quad \Rightarrow \quad 1 = 197.3\,MeV\ fm \quad \Rightarrow \quad 1\ fm = \frac{1}{197.3}\,MeV^{-1} \quad \Rightarrow$$

$$1\ fm = \frac{1}{197.3}\,10^{3}\,GeV^{-1} \qquad (3)$$

However, $$1\text{mb} \;=\; 10^{-3}\,b \;=\; 10^{-3}\,10^{-28}\,m^{2} \;=\; 10^{-31}\left(10^{15}\,fm\right)^{2} \;=\; 10^{-1}\,fm^{2} \quad \Rightarrow$$

$$1\text{mb} = 10^{-1}\,fm^{2} \qquad (4)$$

$$(3)\ (4)\ \Rightarrow\ \mathbf{1mb}\ =\ 10^{-1}\left(\frac{10^3}{197.3}GeV^{-1}\right)^2\ \Rightarrow\ \mathbf{1mb}=0.1(5.068\,GeV^{-1})^2\ \Rightarrow$$

$$\mathbf{1mb}\ =\ 2.569\,GeV^{-2}$$

or

$$\mathbf{1GeV}^{-2}=0.389\,mb$$

II.

$$\hbar c\ =\ 197.3\,MeV\ fm\ =\ 1\ \Rightarrow\ 10^{-15}\,m\times197.3\,MeV\ =\ 1\ \Rightarrow$$

$$\mathbf{1m}\ =\ \frac{1}{10^{-15}\times197.3\,MeV}\ =\ 5.068\times10^{15}\,GeV^{-1}$$

III.

$c\ =\ 3\times10^8\,\frac{m}{sec}$ and if $c\ =\ 1$ we have that

$$1\,sec\ =\ 3\times10^8\,m\simeq3\,10^8\times5\,10^{15}\,GeV^{-1}=1.5\times10^{24}\,GeV^{-1}$$

Problem 2: Show that:

I. The Compton wavelength for an electron is $\lambda_c=\frac{1}{m_e}$
II. The Bohr radius of a Hydrogen atom is $r_{Bohr}=\frac{1}{\alpha\, m_e}$
III. The velocity of an electron in the lowest Bohr orbit is α
IV. Calculate the numerical values for the three expressions above.

where $\alpha=\frac{1}{137}$ is the fine structure constant. The electron mass is $m_e=0.511\mathrm{MeV}$

Solution:

I. The Compton wavelength of the electron is give by: $\lambda_c=\frac{\hbar}{m_e c}$ and if one converts this to a system where $\hbar=c=1$ you get $\lambda_c=\frac{1}{m_c}$. The numerical value for it can be computed as follows:

$$\lambda_c\ =\ \frac{\hbar c}{m_e c^2}\ =\ \frac{197.3\,MeV\ fm}{0.511\,MeV}\ =\ 386\ fm$$

II. The Bohr radius is given by $r_{Bohr}=\dfrac{\hbar}{\alpha\, m_e c}$ and if one converts this to a system where

$\hbar=c=1$ you get $r_{Bohr}=\dfrac{1}{\alpha\, m_e}$ and $r_{Bohr} \; = \; 137\times 386\, fm \; = \; 5.2910^4\, fm$

III. The velocity can be calculated from

$$\frac{m_e v^2}{r_{Bohr}} = \frac{1}{4\pi\epsilon_0}\times\frac{e^2}{r^2_{Bohr}} \qquad \Rightarrow \qquad m_e v^2 = \frac{1}{4\pi\epsilon_0}\times\frac{e^2}{r_{Bohr}} \qquad (1)$$

However the fine structure constant is given by:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137} \text{ which if } \hbar = c = \epsilon_0 = 1 \text{ becomes}$$

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137} \qquad (2)$$

$$(1)(2) \qquad \Rightarrow m_e v^2 = \frac{\alpha}{r_{Bohr}} \Rightarrow v^2 = \frac{\alpha}{r_{Bohr}\times m_e} = \frac{\alpha}{\frac{1}{\alpha\times m_e}\times m_e} = \alpha^2$$

Hence, $\qquad v = \alpha$

<u>Problem 3:</u> Show that due to the fact that the electromagnetic interaction is relatively weak we can use the non-relativistic Schrödinger equation to describe the Hydrogen atoms.

<u>Solution:</u>

The result from part III of problem 2, after the appropriate factor of c is added to match the units are, $\dfrac{v}{c} = \alpha = \dfrac{1}{137}$. This means that the electron velocity is low enough to insure that the relativistic corrections are negligible and can be ignored. Hence, the Schödinger equation can be used to describe the hydrogen atom.

Problem 4: The non-relativistic electromagnetic differential cross-section for scattering a beam of charged particles with charge $q=e$ off a heavy nucleus of charge $Q=Z|e|$ is calculated to be:

$$\frac{d\sigma}{d\Omega}=\left(\frac{Z\alpha}{4E}\right)^2 \sin^{-4}(\theta/2)$$

where $\alpha=\frac{1}{137}$ is the fine structure constant, θ is the scattering angle and E is the beam energy. This calculation assumes that $\hbar=c=1$. Compute differential cross section $\frac{d\sigma}{d\Omega}$ in μb for $E = 1 \ GeV, \theta = 45^0, Z = 12$.

Solution:

This cross-section, like most of the ones you will find in books and High Energy Physics publications, has been calculated in system where, $\hbar=c=1$, and although it is perfectly correct it cannot be used directly to derive numerical results. This should be rather obvious since one expect the cross-section to have units of length-square where this formula has units of inverse-energy-square.

To use this cross-section for computing results one has to first modify the formula and add the appropriate $\hbar$, c factors as needed to get a formula which has the correct units. Since $\hbar c=197.3\,MeV\ fm$ if the above formula it multiplied by $(\hbar c)^2$ the results will have units of length-square and the formula will then be appropriate for preforming calculations with it. The resulting formula is then:

$$\frac{d\sigma}{d\Omega}=\left(\frac{Z\alpha\hbar c}{4E}\right)^2 \sin^{-4}(\theta/2) \qquad \text{(A)}$$

Assume that the energy E is measured in GeV then we can work out the factor:

$$\left(\frac{\hbar c}{E}\right)^2 = \left(\frac{197.3\,10^{-3}\,GeV\ fm}{E}\right)^2 = \frac{0.0389\ \ GeV^2\,fm^2}{E^2} \qquad (1)$$

However, as we have shown before $10\,mb = 1\ fm^2$ (2)

$$(1)(2) \quad \Rightarrow \left(\frac{\hbar c}{E}\right)^2 = \frac{0.0389\ \ GeV^2\,fm^2}{E^2} = \frac{10\times 0.0389\ \ GeV^2\,mb}{E^2} \qquad (3)$$

and (A)(3) $\Rightarrow \frac{d\sigma}{d\Omega} = \frac{10\times 0.0389\times\alpha^2 \quad GeV^2 mb}{16}\left(\frac{Z}{E}\right)^2 \sin^{-4}(\theta/2) \quad \Rightarrow$

$$\frac{d\sigma}{d\Omega} = 1.30\times 10^{-6}\left(\frac{Z}{E}\right)^2 \sin^{-4}(\theta/2)\, GeV^2 mb \quad \Rightarrow$$

$$\frac{d\sigma}{d\Omega} = 1.30\times\left(\frac{Z}{E}\right)^2 \sin^{-4}(\theta/2)\, GeV^2 nb$$

by substituting $E = 1 \ GeV, \theta = 45^0, Z = 12$ we get that

$$\frac{d\sigma}{d\Omega} = 8.70\,\mu b$$

Problem 5: The LHC beam is not continuous but it has a bunch structure. Bunches of particles collide every 25nsec at 13.6 TeV centre of mass energy with a design luminosity[1] (number of particles per sec per cm^2) of 10^{34} $cm^{-2}sec^{-1}$. The total inelastic proton-proton cross section at 13.6 TeV is 70mb.

I. Compute the number of interactions occurring per second as well as every time two bunches collide.

These are called minimum bias interactions and are of no interest (background). The Higgs particle is produced at the LHC mainly through the gluon-gluon production diagram ($gg\rightarrow H^0$) which has a cross section of about 40pb (at low Higgs mass). One of the Higgs discovery channels involves searching for the Higgs decaying to two photons which has a probability (branching ratio) of 0.2×10^{-2}.

II. How many minimum bias events are produced for every Higgs event observed via the two photon channel if one ignores detector effects ?

(an extra correction to these must be applied due to the fact that not all the beam bunches have protons so the stated luminosity corresponds to fewer bunches, but this is beyond the purpose of this course)

1 Over the past 14 years the LHC accelerator has significantly increased the original design luminosity.

Solution:

I. The number of minimum bias events per second can be computed by

$\frac{\Delta N}{\Delta t} = L \times \sigma$ where L is the luminosity and σ the cross-section. Hence,

$$\frac{\Delta N}{\Delta t} = 10^{34}\, cm^{-2}\, sec^{-1} \times 70\, mb = 10^{34} \times 70 \times 10^{-3}\, 10^{-28}\, m^2\, cm^{-2}\, sec^{-1} \quad \Rightarrow$$

$$\frac{\Delta N}{\Delta t} = 10^{34} \times 70 \times 10^{-3}\, 10^{-28}\, 10^{4}\, cm^2\, cm^{-2}\, sec^{-1} \quad \Rightarrow$$

$$\frac{\Delta N}{\Delta t} = 7 \times 10^{8}\, sec^{-1}.$$

So we get 700 million interactions per second. Since the beams collide every 25 nsec then the number of events produced per bunch crossing will be:

$$\Delta N = 7 \times 10^{8}\, sec^{-1} \times 25\, nsec = 17.5 \text{ events every 25 nsec.}$$

II. For Higgs production we follow a similar way but since we are only looking for Higgs decaying to two photons we need to multiply the total Higgs production cross section by the brunching ratio to calculate the cross section for observing a Higgs decaying to two photons. So $\sigma = 40\, pb \times 0.2 \times 10^{-2} = 8 \times 10^{-2}\, pb$. This number can now be used to compute how many Higgs event per second do we expect to observe decaying to two photons.

$$\frac{\Delta N}{\Delta t} = L \times \sigma = 8 \times 10^{-2} \times 10^{-12} \times 10^{-28} \times 10^{4}\, cm^2 \times 10^{34}\, cm^{-2}\, sec^{-1} = 8 \times 10^{-4}\, sec^{-1}$$

The ratio of the two rates is $R = \frac{7 \times 10^{8}}{8 \times 10^{-4}} \sim 10^{12}$.

In other words for every Higgs event which decays into two photons LHC produces 1000 billion minimum bias events. Clearly a very sophisticated event selection procedure must be employed both on-line and off-line to be able to select a single Higgs event in between 1000 billion background events.