Relativistic Kinematics

The Energy Momentum Vector

In the previous lecture Lorentz transformations and covariant notation were reviewed. The purpose of this lecture is to show the students how to use these to compute the various kinematic quantities, such as particle energies, angles and momenta, which are involved in particle physics reactions. This is done by presenting some classical examples that can be found in any textbook as well as by applying relativistic kinematics to demonstrate the Greisen Zatsepin Kuzmin (GZK) limit in High Energy Particle Astrophysics which puzzled physicists in the past and has since been confirmed by high energy cosmic ray experiments. As we do this we also remind the students the meaning of various quantities in particle physics such as the lifetime, the decay width, branching ratio, spin and helicity.

Last time we studied extensively the 4-vector:

$$x^{\mu} = (x^0; \vec{x}) = (ct; \vec{x})$$

and the way it transforms under Lorentz transformations. However, all 4-vectors have the same transformation properties and whatever was stated about x^{μ} applies to any other 4-vector. One of the 4-vectors that we will be using often in particle physics is the energy-momentum 4-vector defined as:

$$p^{\mu} = (p^0; c\vec{p}) = (E; c\vec{p}) = (E; \vec{p}) = (E; p^1, p^2, p^3) = (E; p_x, p_y, p_z) \quad (1)$$

where the 0th component is the energy and the other three the components of the 3-momentum. This quantity transforms as a vector under Lorentz and we will use this for out first example:

Example 1: Consider particle of mass m moving with:

$$\vec{\beta} = \beta \hat{x}_0 = (v/c)\hat{x}_0$$

at the x-direction. Let O' be the reference frame of the particle. In this frame the particle momentum is $\vec{p}' = 0$ and the energy $E' = mc^2$. We want to compute the energy, E, and the momentum of this particle, $\vec{p}$ in the lab frame.

So we want to apply the inverse boost from the particle frame (moving frame) to the Lab frame:

$$p^{\mu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, p^{\nu\,\prime} \Rightarrow$$

$$p^{0} = \gamma\left(p^{0\,\prime} + \vec{\beta}\cdot\vec{p}\,'\right) \Rightarrow p^{0} = E = \gamma\left(mc^{2}+0\right) = m\gamma \qquad (2)$$

$$p_{\parallel} = \gamma\left(p_{\parallel}{}' + \left|\vec{\beta}\right|\cdot p^{0\,\prime}\right) \Rightarrow p_{\parallel} = \gamma\left(0+m\beta\right) = m\gamma\beta \qquad (3)$$

$$p_{\perp} = p_{\perp}{}' = 0$$

In doing this we have shown the formulas for relativistic energy and momentum which students have already seen in special relativity courses. Two formulas that are used frequently are:

$$(2) \Rightarrow \gamma = \frac{E}{m}$$

$$(3) \Rightarrow \beta = \frac{p}{E}$$

Also (2)(3) $\Rightarrow p^{2}+m^{2} = \left(m\gamma\beta\right)^{2}+m^{2} = m^{2}\left(1+\left(\gamma\beta\right)^{2}\right) = m^{2}\gamma^{2}$

Hence,
$$E^{2} = p^{2}+m^{2} \qquad (4)$$

Example 2: Compute $p^{\mu}\cdot p_{\mu}$ in any reference frame:

We know from that the product $p^{\mu}\cdot p_{\mu}$ is a Lorentz invariant so the easiest thing to do is to compute it in the particle rest frame where:

$$p^{\mu} = \left(m;0\right) \Rightarrow p^{\mu}\cdot p_{\mu} = m^{2}$$

Of course one can also compute it in the lab frame using (4)

$$p^{\mu} = \left(E;\vec{p}\right) \Rightarrow p^{\mu}\cdot p_{\mu} = E^{2}-\vec{p}^{\,2} = m^{2}$$

Example 3: Pion Decay: Lifetime and decay length

Consider a negative pion with energy $E=14\,GeV$ which decays in to a muon and a muon anti-neutrino: $\pi^{-} \rightarrow \mu^{-}\bar{\nu}_{\mu}$. If the pion mass is $m_{\pi^{-}} \approx 140\,MeV$ and the pion lifetime is $\tau \approx 2.6\,10^{-8}\,sec$ compute the lifetime of the pion in the lab frame as well as its decay-length.

Solution:

First compute: $$\gamma = \frac{E}{m} = \frac{14\times10^{3}\,MeV}{140\,MeV} = 100$$

The time dilation formula from special relativity gives:

$$t_{LAB} = \gamma\tau = 100\times2.6\times10^{-8} = 2.6\mu sec$$

This can be converted to decay length using:

$$l_{LAB} = \gamma\tau c = 780\,m$$

Decay Rates, Life times and Branching Rations

In the previous example we computed the decay length of the pion and found it to be 780m. This of course does not mean that after 780 m all pions will have decayed. It is well known that particle decays follow the radioactive decay law which states that after time, t, the number of particles that have survived is given by:

$$N(t) = N_0 e^{-\frac{t}{\tau}} \qquad (5)$$

where N_0 is the initial number of particles and τ is the particle lifetime defined in the particle rest frame. This law is the direct consequence of the fact that the decay probability of a particle is a constant and therefore indepentent upon the time that the particle was produced[1]. The effect of the interaction which causes the decay is hidden in the computation of the life time. This is can be calculated using the theory that describes the particular decay. As stated before the lifetime characterises in a sense the type of interaction involved and its coupling strength. For example, particles that decay via the weak interaction such as the $\mu^{\pm}$ or the $\pi^{\pm}$ have by far longer lifetimes than particles that decay via the electromagnetic such as the π^{0} and particles that decay via the strong interaction such as the Δ^{-}, Δ^{0}, Δ^{+}, Δ^{++}, have lifetimes which are far sorter than the previous two categories.

The decay width is defined as the inverse of the life-time and can be computed from the initial and final states of the reaction:

$$\Gamma = \frac{1}{\tau} \sim \left|\langle f|H_{INT}|i\rangle\right|^{2}$$

1 The students are encouraged to start from this and prove the radioactive decay law.

Consider the case where a particle decays to different final states via n-different decay modes: $f_1, f_2, f_3, \dots, f_n$. The total decay width can be computed as:

$$\Gamma_{TOT} = \Gamma_1 + \Gamma_2 + \Gamma_3 + \dots + \Gamma_n$$

To understand the meaning of the decay width we differentiate (5) and we have that:

$$\frac{dN(t)}{dt} = N_0 e^{\frac{-t}{\tau}}(-1)\left(\frac{1}{\tau}\right) = \frac{-N(t)}{\tau} \Rightarrow \frac{1}{N(t)}\frac{dN(t)}{dt} = (-1)\left(\frac{1}{\tau}\right) = -\Gamma$$

Hence, the decay width is simply the decay rate per particle.

The branching ratio, B_i, for a particular decay mode, i, is defined as the percentage of the particles decaying in this mode. Hence,

$$B_i = \frac{\Gamma_i}{\Gamma_{TOT}}$$

Two body decay of a pion

Consider again the decay of a pion to a muon and a muon anti-neutrino:

$$\pi^- \rightarrow \mu^- \bar{\nu}_\mu$$

We will use relativistic energy and momentum conservation to compute the energies and momenta of the final state particles given that:

$$m_\pi \approx 140\,MeV,\; m_\mu \approx 106\,MeV,\; m_\nu \approx 0.$$

It will be shown later in the course that this decay is described by the Feynman diagram in Figure 1. It is a decay the proceeds via the charged current weak interaction which is mediated by a massive charged spin-1 particle, the W-boson.

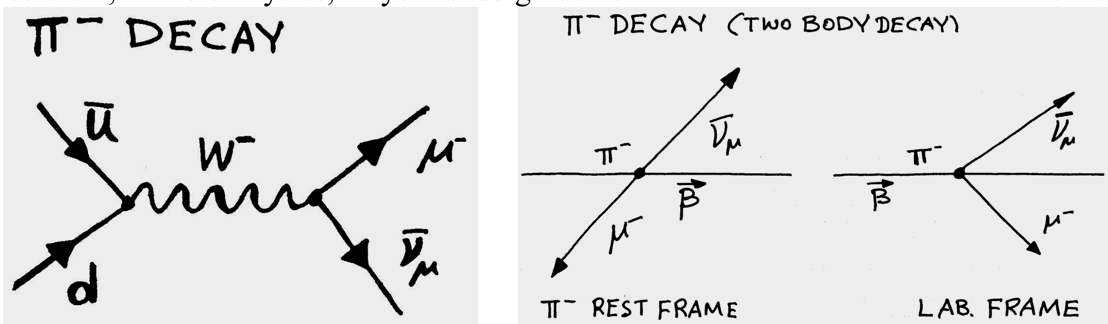


Figure 1: The Feynman diagram of a pion decaying to a muon and a muon anti-neutrino (left). Shown to the right are the decay products of the pion in the pion rest frame and in the laboratory frame.

Let:

$$P_\pi = (m_\pi; 0),\ P_\mu = (E_\mu; \vec{P}_\mu),\ P_\nu = (E_\nu; \vec{P}_\nu)$$

be the energy momentum 4-vectors of the pion, muon and neutrino respectively in the pion rest frame. The decay kinematics are shown in Fig.1 (right). The outgoing particles are back-to-back in the pion rest frame in order to conserve momentum. Energy momentum conservation demands that:

$$P_\pi = P_\mu + P_\nu \qquad (1)$$

Notice that the pion 4-vector is the simplest of all and any 4-vector dot-products that involve this 4-vector are bound to be also simple. So we move the neutrino 4-vector to the left side and square:

$$(1) \Rightarrow \quad (P_\pi - P_\nu)^2 = (P_\mu)^2 \quad \Rightarrow$$

$$P_\pi^2 + P_\nu^2 - 2\mathrm{P}_\pi P_\nu = m_\mu^{\ 2} \quad \Rightarrow$$

$$P_\pi^2 + P_\nu^2 - 2\mathrm{P}_\pi \cdot P_\nu = m_\mu^{\ 2} \quad \Rightarrow$$

$$m_\pi^2 + 0 - 2m_\pi \cdot E_\nu = m_\mu^{\ 2} \quad \Rightarrow$$

$$E_\nu = \frac{m_\pi^{\ 2} - m_\mu^2}{2m_\pi} \quad \Rightarrow$$

$$E_\nu = \frac{(140\,MeV)^2 - (106\,MeV)^2}{2 \times 140\,MeV} \approx 30\,MeV$$

Having computed the energy of the neutrino it is easy to find the momentum since the neutrino is assumed to be massless $|\vec{P}_\nu| = E_\nu \approx 30\,MeV$.

However, as shown in Fig. 1. the neutrino and the muon are emitted back-to-back in the rest frame of the pion and have momenta of equal magnitude because in this frame the initial particle, the pion, has zero momentum. Therefore:

$$\vec{P}_\nu = -\vec{P}_\mu \qquad \Rightarrow \left|\vec{P}_\mu\right| \approx 30\,MeV$$

The muon energy can then be computed as:

$$E = \sqrt{\vec{P}_\mu^2 + m_\mu^2} \approx \sqrt{30^2 + 106^2}\,MeV = 110\,MeV$$

In the pion rest frame the muon has then an kinetic energy of:

$$KE = 110\,MeV - 106\,MeV = 4\,MeV$$

The conclusion is that the energies and momenta of the final state particles in a two body decay are a function of the particle masses and have nothing to do with the nature of the underlying interaction the causes the decay. However, this fact was proven useful in determining that the pion undergoes a two body decay since constant energy of the decay products is a distinct characteristic of a two-body decay.

Shown in Fig. 2 (top) are four examples of emulsion pictures from the discovery of the charged pion in 1947 at the university of Bristol by Powel and Occhialini. Also in Fig. 2 (bottom) is the histogram of the length of the muon tracks which shows that the muon tracks have the same length within experimental resolution. The histogram contains 11 events and the average length 614±8 μm[1].

Fig. 2 demonstrates that the outgoing muon from the pion decay has fixed kinetic energy and of course fixed total energy which proves that the pion undergoes a two-body decay. The explanation why is it so is the following: Muons from stopped pions, as they go through matter, lose energy the same way via a well known process which is called ionization and will be discussed in the next lecture. Hence, they will always travel approximately the same distance from the pion decay point up to the point that they stop and decay themselves if they have the same kinetic energy to begin with. Hence, muon paths of equal length indicate that the muons produced have fixed kinetic energy and this proves that the pion decays into two particles.

Information regarding the nature of the interaction itself can be obtained from the distributions of the decay products and the lifetime of the decay. As we saw before, the lifetime of the particle is directly related to the transition amplitude which can be calculated using interaction hamiltonian which is responsible for the specific decay.

1 D. Perkins, The discovery of the pion in Bristol in 1947, Presented at Varena Conference as part of the comemorations of the 50th anniversary of the pion discovery (CBPF-CS-032/97).

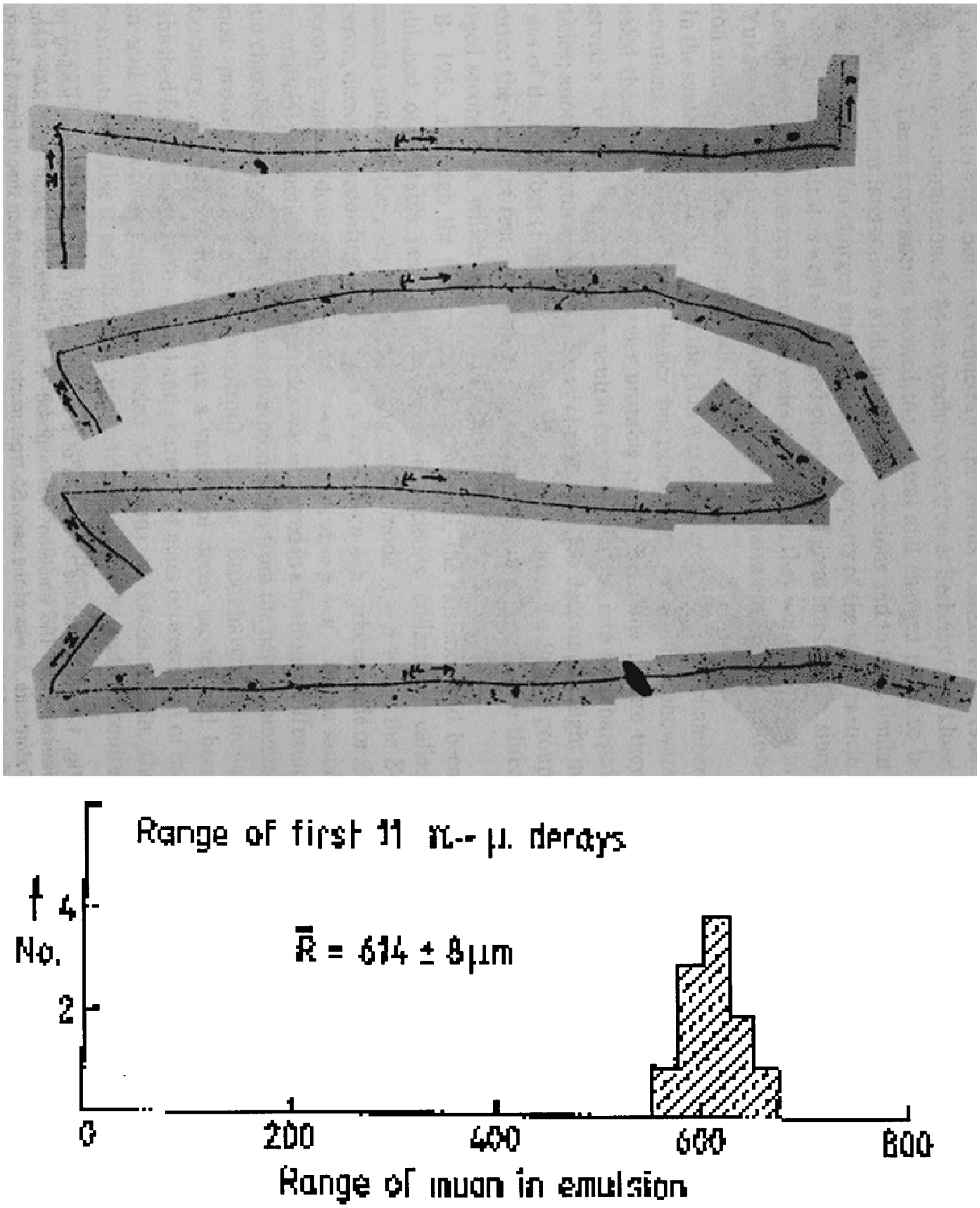


Figure 2: Pion decay signatures in emulsion (top). The pions lose energy and eventually stop in the emulsion material. Stopped pions decay into muons and neutrinos. The pions are the short tracks to the left which decay to a muon moving to the right and an invisible neutrino presumably to the left. The muons lose energy due to ionization and collisions while they move in the emulsion material and stop decaying to an electron and two neutrinos. All muons lose their kinetic energy by traveling the same distance because they have always the same energy to begin with. A histogram of the length of the muon paths in emulsion (bottom) demonstrates that the charged pion decay is a two-body decay.

Isotropic decays

A two body decay is defined to be isotropic if the number of decaying particles per unit solid angle is constant. In other words:

$$\frac{dN}{d\Omega} = C \quad \text{where } \mathbf{C} \text{ is a constant}$$

$$\frac{dN}{d\Omega} = \frac{dN}{d(\cos\theta)\,d\varphi} = C \Rightarrow \frac{dN}{d\theta} = -2\pi C\sin\theta$$

(the minus sign coming from the fact that the cosine is increasing as the angel decreases). Therefore the isotropic decay distribution is flat in $\cos\theta$ but not in θ .

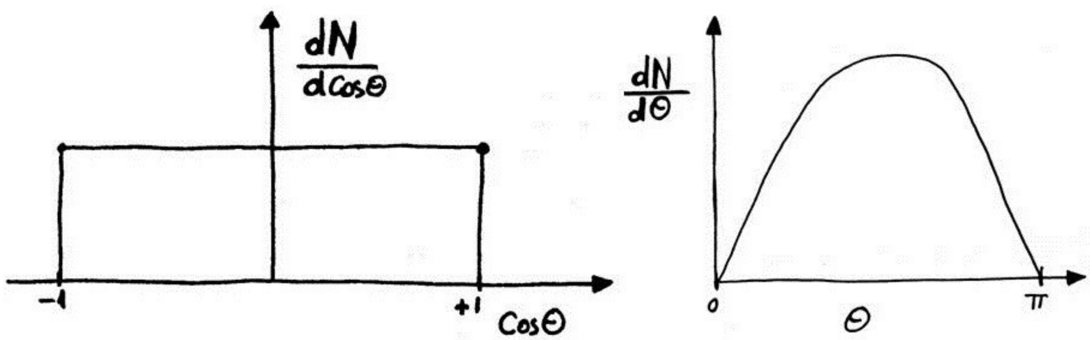


Figure 3: The differential decay rate of events observed as a function of $\cos\theta$ (left) where the distribution is flat and as a function of θ (right) where the distribution is not flat.

Helicity

Define the helicity operator which is the projection of the spin operator on the momentum operator:

$$H = \frac{\vec{s}\cdot\vec{p}}{|\vec{p}|} = \vec{s}\cdot\hat{p} \quad \text{where } \vec{s} = \frac{1}{2}\vec{\sigma}$$

Hence, the eigenvalues of the helicity operator are:

I. $\pm\frac{1}{2}$ for massive fermions. If the fermions are massless then, as we will see later in the course, helicity and handedness are the same. Hence, in this case one can identify these states to left-handed and right-handed fermions.

II. $\pm 1, 0$ for massive spin one objects.

III. ± 1 for massless spin one objects.

Using helicity and angular momentum conservation one can make predictions regarding the spin of the decay products in a two body decay. As an example consider again the decay of a negative pion: $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$, shown in Figure 4. Conservation of total angular momentum can be used to determine the spin of the outgoing negative muon. The spin of the pion has been measured in the 50s and was found to be zero. Also an experimental fact is that only positive helicity anti-neutrinos exist in nature (we will learn more on this when we discuss weak interactions). In the rest frame of the pion the total angular momentum is zero before the decay. The orbital angular momentum of the decay products is also zero. Hence, the spin of the negative muon must be opposite to that of the muon anti-neutrino. Hence, the muon must have also positive helicity.

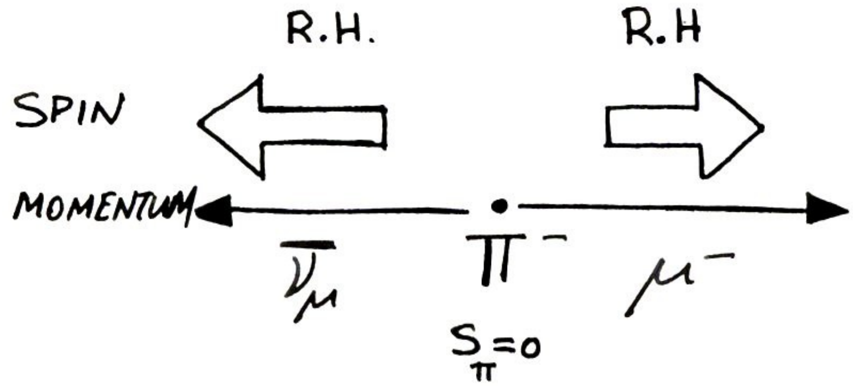


Figure 4:The helicity of the decay products of a negative pion. The anti-neutrino is right-handed because there are no left-handed anti-neutrinos in nature. This forces the muon to be also right-handed because the pion spin is zero.

Reactions

Information about the nature of interactions can also be obtained from the energy thresholds and by measuring scattering cross sections.

Example: The Greisen-Zatsepin-Kuzmin cut-off. This refers to the energy threshold of high energy cosmic rays to interact with the photons from the cosmic microwave background radiation.

I. Considering that the cosmic rays are mostly protons, compute the energy threshold for the reaction $p+\gamma \rightarrow \pi+p$ where a cosmic ray proton collides with a cosmic microwave background (cmb) photon ($3\ K^0$ or $E_\gamma = 1.4\times10^{-3} eV$) to produce a pion and a proton. This is the famous Greisen-Zatsepin-Kuzmin (GZK) cut-off.

II. The spectrum of the observed cosmic ray interactions by the **HiRES**[1] and **AGASA** experiments as shown in Fig. 5. In this plot is seems that the **HiRES** data support the GZK cut-off whilst the **AGASA** data seem to violate it. This had created the GZK puzzle. The question that puzzeled astrophysicits for years was the following. If the **AGASA** data were correct, where do the high energy cosmic rays come from which have energy above the GKS limit ?. Assuming that the cross section above threshold for this reaction is $\sigma_{\gamma p} = 200\,\mu B$ and that the cosmic microwave background photon density is $\rho = 550\ \gamma/cm^3$ compute the mean-free-path for the comic ray protons and show that they could not possibly come from outside our galaxy.

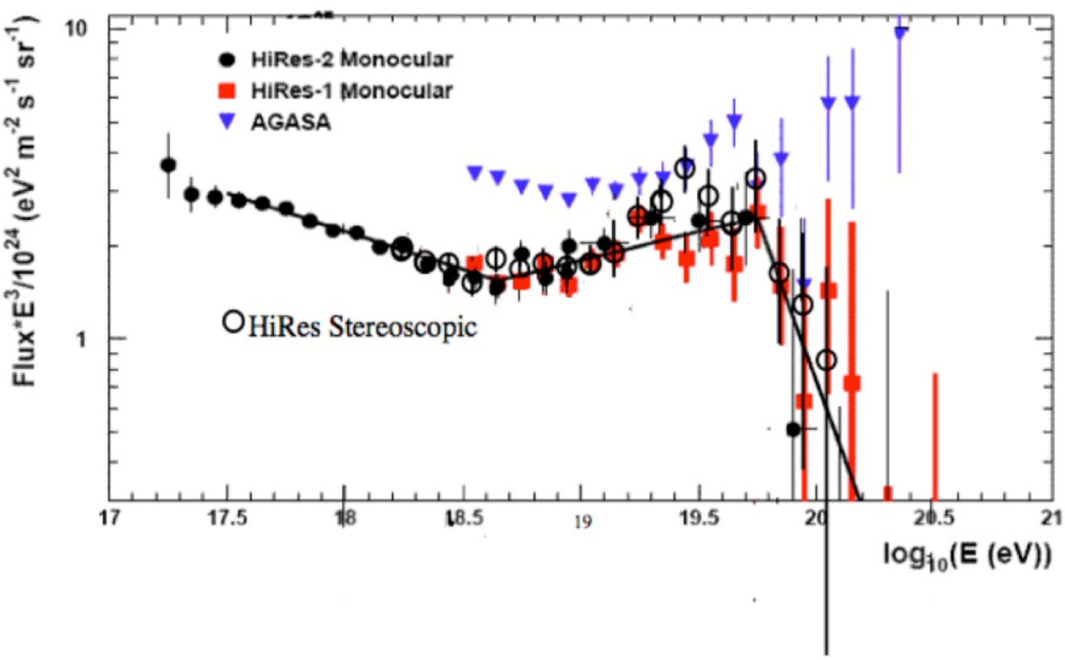


Figure 5: *Measurements of the flux of cosmic rays versus energy from HiRES and AGASA. The HiRES data support a cutoff at about 10^{19} eV where the AGASA data show a flat scetrum which goes beyond 10^{19} eV.*

Solution:

$$P_\gamma + P_p = P_\pi + P'_p \Rightarrow (P_\gamma + P_p)^2 = (P_\pi + P'_p)^2 \Rightarrow$$

$$m_\gamma^2 + m_p^2 + 2P_\gamma P_p = m_\pi^2 + m_p^2 + 2P_\pi P'_p \quad (1)$$

1 *Sokolsky; for the HiRes Collaboration (2010). "Final Results from the High Resolution Fly's Eye (HiRes) Experiment". Nuclear Physics B: Proceedings Supplements. 212–213: 74–78.* arXiv:1010.2690.

The threshold for this reaction is defined to be the incident proton energy where the outgoing proton and pion are produced with zero kinetic energy whist the incoming photon and proton collide head-on. Therefore:

$$(1) \Rightarrow m_p^2+2E_\gamma E_p-\vec{P}_\gamma\cdot\vec{P}_p=m_\pi^2+m_p^2+2E_\pi E'_p-2\vec{P}_\pi\cdot\vec{P'}_p$$

The proton mass terms cancel and in any case at the energies involved the energy of the proton is equal to its momentum in GeV. Hence,

$$4E_\gamma E_p=m_\pi^2+2E_\pi E'_p-2\vec{P}_\pi\cdot\vec{P'}_p \Rightarrow$$

(the momenta of the out-going pion and proton are zero so only the energies survive as masses)

$$E_p = \frac{m_\pi^2+2m_\pi m_p}{4E_\gamma} \Rightarrow$$

$$E_p = \frac{(0.140)^2+2\times0.140\times0.940}{4\times1.4\times10^{-3}\times10^{-9}}\approx5\times10^{19}\,eV$$

This threshold is the **Greisen Zatsepin Kuzmin** cut-off and tells us that protons with energy above this threshold will interact with the microwave background radiation. Hence, this reaction can occur for the highest energy range particles, above the GZK threshold. As shown in Fig. 5 the AGASA[1] data were above this limit and this created the puzzle as to where from these cosmic rays came from.

The mean free path for interaction is given by:

$$\Lambda = \frac{A}{N_A\times\rho\times\sigma} = \frac{\Delta V}{\sigma}$$

Where **A** is the atomic weight, $\mathbf{N_A}$ the Avogadro number, ρ is the density and σ is the cross section of the interaction. In the second expression the quantity ΔV represents the elementary volume occupied by a target quantum, in this case a CMB photon. Since the cross section and the number of photons per square centimeter is given we can use the second expression to compute the mean fee path:

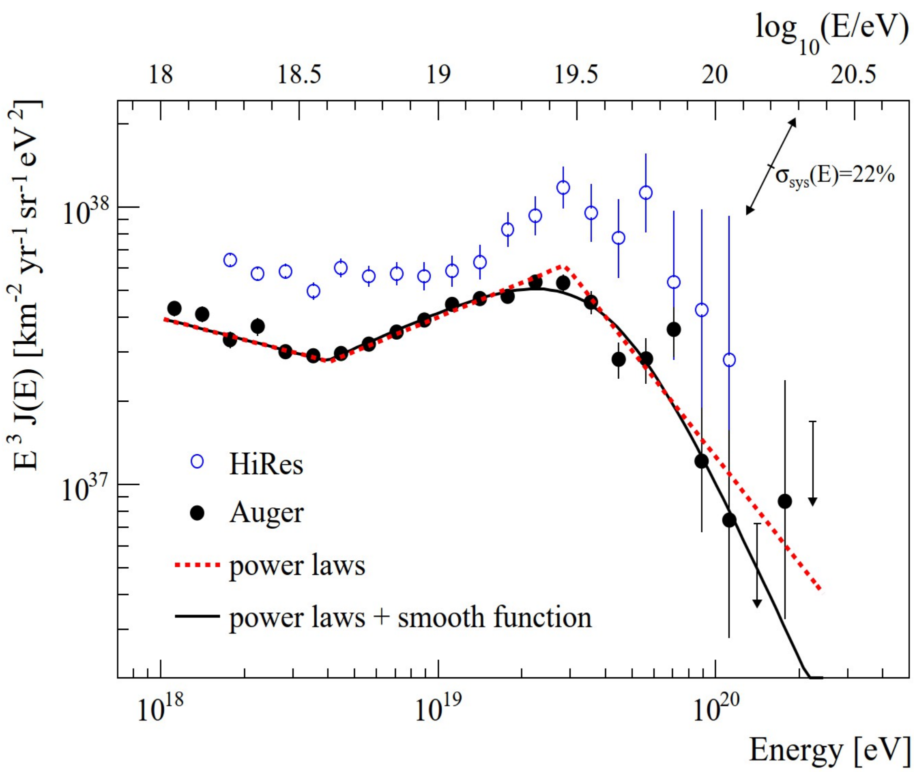
$$\Lambda = \frac{1}{550\times10^{+6}m^{-3}\times200\times10^{-6}\times10^{-28}m^2} = 9.1\times10^{22}m \Rightarrow$$

1 S. Yoshida *et all, "The cosmic ray energy spectrum above 3 10^8 eV measured by the Akeno Giant Air Shower array, Astroparticle Physics 3 (1995), 105-123*

$$\Lambda = 3\,Mpc \qquad (\mathbf{1pc} = 3 \times 10^6 m)$$

Hence, the mean free path due to this effect is smaller than intergalactic distances and the likelihood is that protons produced at distances larger than this will interact before the get to us and cannot be observed close to earth. This means that the high energy events from AGASA must have been produced *inside our galaxy* and the big question at the time was where do these particles came from. The threshold energy we calculated is the famous Greisen, Zatsepin, Kuzmin (GZK) cut off. However, since then data from Auger[1] and HiRES confirm thatr there is indeed a cutoff at the GZK threshold. These data are shown in Fig. 6. So the AGASA data were not confirmed by these two experiments which confirmed beyond any doubt the GZK cut-off. **The conclusion of this investigaton is that there are no sources of high energy cosmic rays beyond the GZK limit within our galaxy.**

1 *The Pierre Auger Collaboration (2010). "Measurement of the energy spectrum of cosmic rays above 10^{18} eV using the Pierre Auger Observatory". Phys. Lett. B.* ***685*** *(4–5): 239–246. arXiv:1002.1975.*



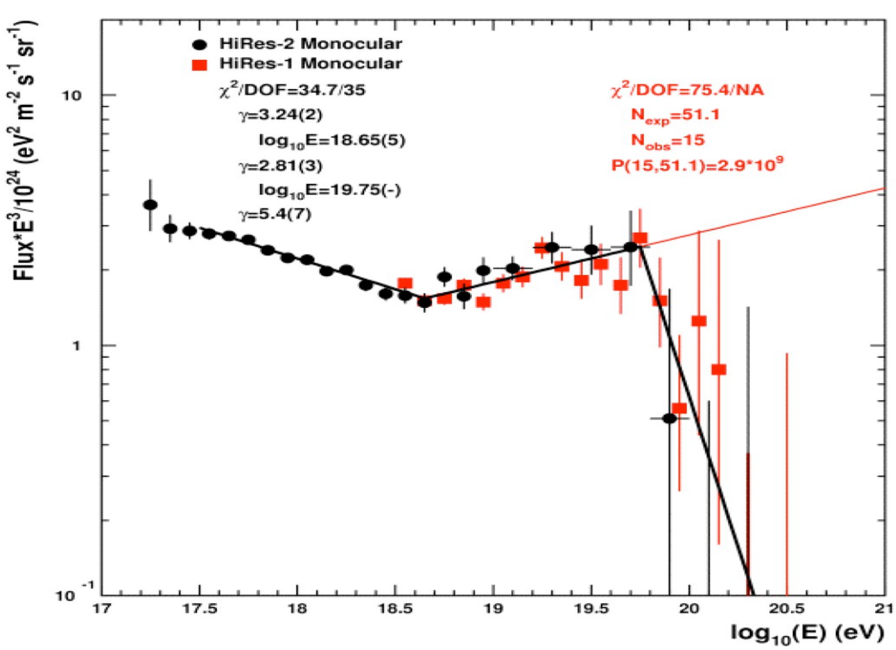


Figure 6: The Auger and HiRES data spectrum support the GZK limit.