Tensor notation

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Tensor notation in three dimensions:

We present here a brief summary of tensor notation in three dimensions simply to refresh the memory of the reader and provide a smooth introduction to the relativistic tensor notation which follows.

Rotations: The equation for rotating the coordinates (x, y) of a vector in two dimensions by an angle θ clockwise is given by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where (x', y') are the coordinates of the rotated vector. This can be generalized in three dimensions by the equation:

$$X_i' = \sum_{j=1}^{j=3} R_{ij} X_j \qquad (1)$$

where R_{ij} is the rotation matrix and X_j, X_i' are the initial vector and the vector after rotation in three dimensions.

Since most of these equations involve summations we will drop repeated summation symbols and instead we will assume summation whenever we have two indices which are repeated. This is called the **Einstein convention**. In the case of (1) the index j is repeated so it means summation over j and we can drop the summation sign. So equation (1) becomes simply:

$$X_i' = R_{ij} X_j$$

Equation (1) is the transformation of a **vector** under rotations.

A fundamental property of rotations is that they leave the magnitude of a vector invariant. This leads to the following relationship:

$$X_i' X_i' = R_{ij} X_j R_{il} X_l = R_{ij} R_{il} X_j X_l = X_j X_j \qquad (2)$$

Which means that:

$$R_{ij} R_{il} = \delta_{il} \qquad (3)$$

Which is indeed a fundamental property of the rotation matrices.

In a similar way that relation (1) defines the object which is called vector because it transforms according to (1) under rotations, we can define more complicated objects which are called tensors. For example a **rank-2 tensor** is an object with two indices which transforms under rotations according to:

$$T'_{ij} = R_{il} R_{jm} T_{lm}$$

This can of course be generalized to **n-rank tensors** as:

$$T'_{ij.....n} = R_{il} R_{jm} ...... R_{na} T_{lm.....a}$$

A polar vector in 3 dimensions (3-D) is an object which, given a coordinate frame, can be defined as

$$\vec{a} = (a_x, a_y, a_z) = (a_1, a_2, a_3)$$

and transforms as a vector under rotations. It is usual in text books that the standard (x, y, z) coordinate indices are replaced by the (1, 2, 3) indices. Furthermore, parity inverts all the components of a polar vector, such that:

$$P\vec{a} = (-a_x, -a_y, -a_z) = (-a_1, -a_2, -a_3)$$

Hence, a <u>polar vector</u> is an object which transforms as a vector under rotations and all its components change sign under parity.

The dot-product of two vectors is defined as:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i \qquad i=1,2,3$$

As seen at the last step we have used the Einstein convention under which repeated indices indicate summation.

The dot-product is invariant under rotation and parity transformations. Hence, it is a <u>scalar</u>. Lets see why is that. Suppose we have two vectors $\vec{a}', \vec{b}'$ which have been produced by rotating two other vectors $\vec{a}, \vec{b}$. In other words we have that:

$$a'_i = R_{il} a_l \qquad (3)$$

$$b'_i = R_{im} b_m \qquad (4)$$

Using (3) an (4) we have that:

$$\vec{a}'\cdot\vec{b}' = a'_i b'_i = R_{il} R_{im} a_l b_m = \delta_{lm} a_l b_m = a_l b_l = \vec{a}\cdot\vec{b}$$

Hence, the dot-product is invariant under rotations. Obviously since both vectors change sign under parity the dot-product will remain invariant under parity.

Similarly one can rewrite in this notation operators which are formed by taking dot-products. For example

$$\vec{\nabla}\cdot\vec{A} = \partial_i A_i \quad \text{or} \quad \vec{\nabla}^2 = \partial_i \partial_i$$

To define the cross product we first need to define the Levy-Civita tensor:

$\epsilon^{ijk} = +1$ if (i, j, k) = (1, 2, 3) or (2, 3, 1) or (3, 1, 2) (even permutations)

$\epsilon^{ijk} = -1$ if (i, j, k) = (2, 1, 3) or (3, 2, 1) or (1, 3, 2) (odd permutations)

$\epsilon^{ijk} = 0$ if any indices are the same.

The cross product is normally defined as:

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$$

where $\hat{x},\ \hat{y},\ \hat{z}$ are the unit vectors in the x, y, z directions. This definition translated to tensor notation reads as:

$$(\vec{a}\times\vec{b})_i = \epsilon_{ijk} a_j b_k$$

where the index i indicates the ith component of the cross product. Lets see how the cross product transforms under rotations:

Let $a_i b_i c_i$ be the components of three polar vectors in a coordinate system **O** which are related by

$$c_i = (\vec{a}\times\vec{b})_i = \epsilon_{ijk} a_j b_k \qquad (5)$$

and $\tilde{a}_i \tilde{b}_i$ be the components of the two polar vectors in a system **O'** which is rotated relative to **O**. In other words:

$$a_i = R_{ij}\tilde{a}_j \quad b_i = R_{ij}\tilde{b}_j$$

The question we would like to answer is that given that $\vec{a},\vec{b}$ transform under rotations as vectors what can we conclude about the third object which is related to them via (5) ? If we can show that $c_i = R_{ij}\tilde{c}_j$, then this means that the cross product of two vectors transforms also as a vector:

$$c_i=\varepsilon_{ijk}\,a_j\,b_k=\tilde{\varepsilon}_{\alpha\beta\gamma}\,R_{i\alpha}\,R_{j\beta}\,R_{k\gamma}\,R_{jl}\,R_{km}\,\tilde{a}_l\,\tilde{b}_m=\tilde{\varepsilon}_{\alpha\beta\gamma}\,\delta_{\beta l}\,\delta_{\gamma m}\,R_{i\alpha}\,\tilde{a}_l\,\tilde{b}_m=R_{i\alpha}\,\tilde{\varepsilon}_{\alpha\beta\gamma}\,\tilde{a}_\beta\,\tilde{b}_\gamma\Rightarrow$$

$$c_i=R_{i\alpha}(\vec{\tilde{a}}\times\vec{\tilde{b}})_\alpha \quad\Rightarrow\quad c_i=R_{i\alpha}\,\tilde{c}_\alpha$$

In other words **the cross-product of two vectors transforms as a vector under rotations. However, <u>under parity it does not change sign like a vector</u>. Hence, it is called <u>axial vector</u>.**

The quantity $\vec{c}\cdot(\vec{a}\times\vec{b}) = c_i\epsilon_{ijk}\,a_j\,b_k$ as a dot product remains invariant under rotations. **However, it changes sign under parity hence it is called <u>pseudoscalar</u>.** Two examples are presented to demonstrate how powerful is the tensor notation:

Example 1: Show that $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \partial_i(\epsilon_{ijk}\partial_j A_k) = \epsilon_{ijk}\partial_i\partial_j A_k = 0$$

The last step results from the summation an antisymmetric tensor, ϵ_{ijk} , with a symmetric one, $\partial_i\partial_j$.

Example 2: Show that $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}$

$$\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right]_i = \epsilon_{ijk}\partial_j\left(\vec{\nabla}\times\vec{A}\right)_k = \epsilon_{ijk}\partial_j\left(\epsilon_{klm}\partial_l A_m\right) = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m \quad (1)$$

Using the identity:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm}-\delta_{im}\delta_{jl} \quad (2)$$

we have:

$$(1)(2) \Rightarrow \left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right]_i = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m = \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\partial_j\partial_l A_m \Rightarrow$$

$$\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right]_i = \partial_i\left(\partial_l A_l\right)-\partial_j\partial_j A_i = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{\nabla}^2\vec{A}$$

Example 3: Show that the fully contracted product of a symmetric tensor, S_{ij} with an antisymmetric tensor A_{ij} vanishes.

So we have that $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$. Next we form the fully contracted product of the two as follows.

$$S_{ij} A_{ij} = \frac{1}{2}\left(S_{ij} A_{ij} + S_{ji} A_{ji}\right) = \frac{1}{2}\left(S_{ij} A_{ij} - S_{ij} A_{ij}\right) = 0$$

Introduction to the Covariant Notation:

All equations presented in this course are Lorentz invariant (relativistic invariant) and the notation we use to write down Lorentz invariant equations is called covariant notation. Use Lorentz invariant equations for two reasons:

- The first reason addresses the need for relativistic invariant theories. One of the most fundamental principles in physics is the fact that all equations and subsequently their predictions should be invariant with respect to the frame of reference. Otherwise physics would not be an objective science.
- The second reason is a practical one but no less important and it addresses the need for using special relativity in High Energy Physics (HEP): At the energies that the HEP deals most particles have velocities approaching the velocity of light so they need to be treated according to the formulae of the Special Relativity.

So we start by introducing the covariant notation and the Lorentz transformations within this notation. Every point in space-time can be represented by a ***contravariant 4-vector*** defined as:

$$x^\mu = \left(x^0 ; \vec{x}\right) = \left(ct ; \vec{x}\right) \qquad (1)$$

The vector index in (1) is running between $\mu = 0, 1, 2, 3$ and the 4-vector is defined explicitly as:

$$x^\mu = \left(x^0 ; x^1, x^2, x^3\right) = \left(ct ; x, y, z\right)$$

where **t** is the time and **x, y, z** are the three space coordinates. Greek indices denote always space-time variables whilst latin indices denote always 3-dimensional variables.

A more rigorous definition of a 4-vector in terms of its transformation properties will be given later once the Lorentz transformation has been defined in covariant notation.

Define to dot-product of two contravariant vectors to be the matrix product:

$$x \cdot y = x^{\mu} g_{\mu\nu} y^{\nu} \qquad (2)$$

where :

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (3)$$

is the metric tensor of the flat Minkowski space. Several books including Perkins use several different metric definitions. The end result does not of course change but it does create confusion. The definition in (3) is the most popular in HEP and Relativity books and this is what we will follow throughout this course.

It is worth noticing that:

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(1) We have used the Einstein convention where two indices that are the same indicate summation. This of course implies that in a given expression one cannot have more than 2 indices which are the same. Sometimes we refer to this summation as index contraction because the summed indices disappear at the end. This is also why summed indices can be re-named at wish: Since they disappear you can change the index to whatever symbol is convenient for the calculation.

(2) The metric indices are subscripts in this case whilst the contravariant vector indices are <u>always</u> superscripts. This is not an accident will become clear later why we do that.

In a more explicit way the definitions (2) (3) mean that:

$$x \cdot y = x^{\mu} g_{\mu\nu} y^{\nu} = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix} = x^0 y^0 - \vec{x} \cdot \vec{y}$$

Using the definitions the dot-product of a 4-vector with itself is:

$$(1)(2)(3) \Rightarrow x \cdot x = \left(x^0\right)^2 - \vec{x}^2 = \left(ct\right)^2 - \vec{x}^2$$

Hence, t**he dot-product of a 4-vector with itself is a relativistic invarian**t since a spherical light wave should look spherical in every coordinate frame (the speed of light is the same in all intertial reference frames). It will be shown later that the dot-product of *any* 4-vectors is relativistic invariant.

Although we can do all of calculations using contravariant vectors, we will need to include always the metric when we take dot-products. This is inconvenient and this is why we need to introduce the *covariant vectors*:

A ***covariant 4-vector*** is defined as:

$$x_\mu = (x^0; -\vec{x}) = (ct; -\vec{x}) \,. \tag{4}$$

Notice that except the space vector sign which is negative we have also changed the 4-vector index from superscript to subscript.

Covariant tensors are always associated with subscripts in this notation. The dot-product can now be defined from the covariant and contravariant vectors without the explicit use of the metric as:

$$x \cdot x = x_\mu x^\mu = (ct)^2 - \vec{x}^2 = g_{\mu\nu} x^\nu x^\mu \tag{5}$$

Equations (4) and (5) imply that:

$$x_\mu = g_{\mu\nu} x^\nu$$

As seen here **the metric can be used to lower an index and convert a contravariant vector to a covariant vector.** The opposite is also true if one defines the metric to be the same for both covariant and contravariant indices:

$$g^{\mu\nu} = g_{\mu\nu}$$

and in this case the metric can be used to rise an index:

$$x^\mu = g^{\mu\nu} x_\nu$$

and convert a covariant 4-vector to a contravariant 4-vector.

In this notation one can define the Kroneker delta as:

$$\delta^{\mu}{}_{\nu} = g^{\mu\rho} g_{\rho\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since it must be that

$$\delta^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x^{\nu}}$$

we are driven to the convention where contravariant indices in the denominator become covariant indices in the numerator and visa versa. This can be seen also in the case where

$$g^{\mu\nu} = \frac{\partial x^{\mu}}{\partial x_{\nu}}$$

Clearly in this notation we have that $g_{\mu\nu} g^{\nu\mu} = 4$.

Contravariant and covariant derivatives are then defined as:

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial x^{0}} ; \vec{\nabla}\right)$$

and

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial x^{0}} ; -\vec{\nabla}\right)$$

Lorentz Transformations

Our definition of a contravariant 4-vector in (1) whilst easy to understand is not the whole story. **A contravariant 4-vector is an object defined as $x^{\mu} = \left(x^{0} ; \vec{x}\right)$ that transforms as a vector under Lorentz transformations.** That is:

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad (6)$$

where x^ν is the 4-vector in a frame **O** and $x^{\mu'}$ a 4-vector in the reference frame **O'** related to x^ν by the Lorentz transformation $\Lambda^\mu{}_\nu$. The matrix $\Lambda^\mu{}_\nu$ is in the general case a complex object which can represent a mixture of Lorentz boosts and 3-D rotations. However, in the case of a pure Lorentz boost by $\vec{\beta} = \beta \hat{x}_0 = (v/c)\hat{x}_0$ in the x-direction, it assumes a form which is familiar from special relativity texts which can be written as:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (7)$$

where $\vec{v}$ is the relative velocity of the frame **O'** with respect to **O**.

$$(6)(7) \Rightarrow x^{\mu'} = \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \Rightarrow$$

$$\begin{aligned} x^{0'} &= \gamma\left(x^0 - \vec{\beta}\cdot\vec{x}\right) \\ x^{1'} &= \gamma\left(x^1 - \beta x^0\right) \\ x^{2'} &= x^2 \\ x^{2'} &= x^2 \end{aligned} \qquad (8)$$

The four equations in (8) represent a Lorentz boost for $\vec{\beta} = \beta \hat{x}_0$.

Now that the general form of a Lorentz transformation has been defined under (6) we can investigate the consequences of the relativity constraint namely that the product of a 4-vector with itself must be a Lorentz invariant:

$$x' \cdot x' = g_{\mu\nu} x^{\mu\prime} x^{\nu\prime} = g_{\mu\nu} \Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta x^\beta \Rightarrow$$

$$x' \cdot x' = \left(g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\right) x^\alpha x^\beta = x \cdot x \Rightarrow \left(g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\right) = g_{\alpha\beta} \Rightarrow$$

$$\left(\Lambda^\mu{}_\alpha \Lambda_{\mu\beta}\right) = g_{\alpha\beta} \quad \Rightarrow \quad g_{\alpha\beta} g^{\beta\sigma} = \left(\Lambda^\mu{}_\alpha \Lambda_{\mu\beta}\right) g^{\beta\sigma} \quad \Rightarrow$$

$$\Lambda^\mu{}_\alpha \Lambda_\mu{}^\sigma = \delta_\alpha{}^\sigma \qquad (9)$$

This is a basic property of the Lorentz transformations in general. It is as simple exercise to show that (7), which is a particular type of Lorentz transformation is consistent with (9) which holds for a whole family of Lorentz transformations.

Now we are ready to show that the product of any two 4 vectors is a Lorentz invariant quantity:

$$(8)(9) \quad \Rightarrow \quad x' \cdot y' = x^{\mu}{}' y_{\mu}{}' = x^{\mu}{}' y^{\nu}{}' g_{\mu\nu} = \Lambda^{\mu}{}_{\alpha} x^{\alpha} \Lambda^{\nu}{}_{\beta} y^{\beta} g_{\mu\nu} \Rightarrow$$

$$x' \cdot y' = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} g_{\mu\nu} x^{\alpha} y^{\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda_{\mu\beta} x^{\alpha} y^{\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda_{\mu}{}^{\beta} x^{\alpha} y_{\beta} \Rightarrow$$

$$x' \cdot y' = \delta_{\alpha}^{\beta} x^{\alpha} y_{\beta} = x \cdot y$$

Hence, t**he dot-product of any 4-vectors is a relativistic invariant.** Notice that this is always the product of a covariant with a contravariant 4-vector. So when we contract covariant with contravariant indices the results of the summation is always going to be a Lorentz invariant quantity which is what we usually want to have in physics. It is for this reason that when we use Einstein's convention to sum over indices we always have two same indices but one is covariant and the other contravariant.

Next we can calculate the **inverse boost** of (6), $(\Lambda^{-1})^{\alpha}{}_{\rho}$. We start with (9) and multiply both sides by $(\Lambda^{-1})^{\alpha}{}_{\rho}$:

$$\Lambda^{\mu}{}_{\alpha} \Lambda_{\mu}{}^{\beta} = \delta_{\alpha}^{\beta} \Rightarrow \Lambda^{\mu}{}_{\alpha} (\Lambda^{-1})^{\alpha}{}_{\rho} \Lambda_{\mu}{}^{\beta} = \delta_{\alpha}^{\beta} (\Lambda^{-1})^{\alpha}{}_{\rho} = (\Lambda^{-1})^{\beta}{}_{\rho} \qquad (10)$$

However since $(\Lambda^{-1})^{\alpha}{}_{\beta}$ is the inverse we have that:

$$\Lambda^{\mu}{}_{\alpha} (\Lambda^{-1})^{\alpha}{}_{\rho} = \delta^{\mu}{}_{\rho} \qquad (11)$$

$$\delta^{\mu}{}_{\rho} \Lambda_{\mu}{}^{\beta} = (\Lambda^{-1})^{\beta}{}_{\rho} \Rightarrow \Lambda_{\rho}{}^{\beta} = (\Lambda^{-1})^{\beta}{}_{\rho} \qquad (12)$$

The Lorentz transformation for vectors (6) can be extended for rank-2 tensors. **Hence, a rank-2 tensor is an object that under Lorentz transformations transforms as:**

$$T^{\alpha\beta}{}' = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} T^{\mu\nu} \qquad (13)$$

It is then easy to show that the sum:

$$T^{\alpha\beta} x_\alpha x_\beta \qquad (14)$$

is a Lorentz invariant quantity if $T^{\alpha\beta}$ transforms according to (13) and x_α, x_β transform according to (6). Obviously (13) can be extended to any rank-n tensor. In that case we have from (14) that any quantity where all the indices have been summed up (fully contracted indices) will be a relativistic/Lorentz invariant.

Parity transformation properties of 4-vectors:

In a similar way as in the 3-D case, a combination of the Lorentz and Parity properties can be used to classify the various fields and currents we use in Particle Physics as follows:

- Quantities which are invariant under Lorentz transformations are called ***scalars (S)*** if they are even under parity. If they are odd under parity they are called ***pseudo-scalars (P)***.

- Similarly, by definition, the 4-vectors which transform under parity as:

  $$V^\mu = (V^0; \vec{V}) \rightarrow V_P^\mu = (V^0; -\vec{V})$$

  are commonly referred to as ***polar 4-vectors (V).***

- Finally, the category of Lorentz vectors which transform under parity as:

  $$A^\mu = (A^0; \vec{A}) \rightarrow A_P^\mu = (-A^0; \vec{A})$$

  are called ***axial 4-vectors (A).***

Maxwell's Equations in Covariant Notation

One can see how powerful can be the covariant notation by writing Maxwell's equations in a covariant form. Start with Maxwell's equations in the standard 3D notation:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Define the electromagnetic 4-vector potential and 4 vector current to be

$$A^{\mu} = (\Phi;\vec{A}) \qquad J^{\mu} = (\rho;\vec{J}) \tag{15}$$

where $\Phi,\vec{A}$ are the electromagnetic scalar and vector potential and $\rho,\vec{J}$ are the charge and current densities. The Electromagnetic Field tensor can be defined as a function of the 4-vector potential as:

$$F^{\mu\nu} = \partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu} \tag{16}$$

which in terms of the electric and magnetic fields can be written as:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{17}$$

The field tensor can also be written as:

$$F_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{18}$$

Another form of the field tensor the dual, $\tilde{F}^{\mu\nu}$, can be defined by:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{19}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is defined to be : $\epsilon^{\mu\nu\alpha\beta} =$ zero if any indices are the same, -1 for odd permutations and +1 for even permutations of the indices.

Using (15)(16)(17)(18)(19) the four Maxwell Equation can be written as two covariant equations:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{20}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{21}$$

The first one corresponds to the first and the fourth Maxwell equations which contain sources and the second one corresponds to the other two.