Lecture 12 30/03/24

Lagrangians in Classical and Quantum Physics

In the previous lectures we presented a relativistic quantum description of the free spin-0 and spin-½ particles via the Klein-Gordon and the Dirac equations respectively. We also discussed the Maxwell equations which describe a massless spin-1 particle the photon. In this lecture we will present an equivalent formulation to describe the same particles based on Lagrangians from which one can extract the same equations of motion in an analogue way as done in Classical Mechanics. Both formulations describe identical physical processes and none of them provides more information than the other. The advantage of starting from the Lagrangian is that this way symmetries become apparent. Furthermore, one can introduce Quantum Field Theories more naturally starting from the Lagrangian formulations.

We will start with a short review of the the Lagrangian formulation in Classical Mechanics and will discuss the role that space time symmetries play in constructing conserved quantities.

The Principle of Least Action:

In Classical Mechanics, equations of motion can be extracted using the principle of Least Action. The action of a given physical system is defined as

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$

where $L(q_i, \dot{q}_i, t)$ is the Lagrangian which is a function of $q_i = q_i(t)$ and $\dot{q}_i = \frac{dq_i(t)}{dt}$, the generalized coordinates and velocities and time. The Lagrangian is defined as

$$L = T - V$$

where T and V are the kinetic and potential energies of the system respectively. The index i runs between $1 \leq i \leq N$ where N is the number of degrees of freedom of the system under consideration.

According to the Principle of Least Action the equations of motion can be extracted by considering path variations such as

$$q_i(t) \rightarrow q_i'(t) = q_i(t) + \delta q_i(t)$$

under the condition that the generalized coordinate variation at the path ends vanishes

$$\delta q_i(t_1) = \delta q_i(t_2) = 0$$

and by requiring that the action integral has a minimum

$$S = S_{min} \Rightarrow \delta S = 0$$

The demand that the action has a minimum implies that

$$\delta S = \int_{t_1}^{t_2} \delta L(q_i, \dot{q}_i, t) = 0 \Rightarrow$$
$$\delta S = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] = 0 \Rightarrow$$
$$\delta S = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right] = 0 \Rightarrow$$
$$\delta S = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] = 0 \Rightarrow$$
$$\delta S = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] + \sum_i \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} = 0 \Rightarrow$$

However, due to the fact that the path variations vanish at the ends of the path we have that

$$\sum_i \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} = 0$$

Finally we have that

$$\int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] = 0 \Rightarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

These are the **Euler-Lagrange** equations which are used to extract the equations of motion from the Lagrangian of a given system. Since the variation of the generalized coordinates vanish at the beginning and the end of the path, the Lagrangian of a system is not uniquely define and we can always add a total derivative $df(q_i(t), t)/dt$ to the Lagrangian without any effect to the equations of motion. This is because the action will changes as

$$S \rightarrow S + f(q_i(t_2), t) + f(q_i(t_1), t)$$

Example 1: Derive the equation of motion for a free particle moving at the positive x-direction.

The kinetic energy of such a particle is given by

$$T = \frac{1}{2} m \dot{x}^2$$

and since the particle is free the potential energy vanishes

$$V = 0$$

Hence, the Lagrangian is given by

$$L = \frac{1}{2} m \dot{x}^2$$

and

$$\frac{\partial L}{\partial x} = 0 \ , \quad \frac{\partial L}{\partial \dot{x}} = m \dot{x}$$

Therefore

$$\frac{d}{dt}(m \dot{x}) + 0 = 0 \Rightarrow m \ddot{x} = 0$$

which tells us that the particle is moving freely on the x-axis and has an acceleration which is equal to zero. Therefore, it has a constant velocity which can be determined by the initial conditions.

Example 2: The kinetic and potential energies of a simple harmonic oscillator oscillating on the x-axis are

$$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{1}{2} k x^2$$

Extract the equations of motion of the simple harmonic oscillator.

The Lagrangian is given by $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$

Therefore, $$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \frac{\partial L}{\partial x} = -kx$$

Hence,

$$m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{k}{m}x = 0 \Rightarrow \ddot{x} + \omega^2 x = 0$$

This is the equation of the simple Harmonic Oscillator.

Conservation Laws as Consequence of the Symmetries of the Lagrangian

Emmy Noether's theorem states that continuous symmetries of the Action/Lagrangian lead to conservation laws and conserved quantities. In this section we will consider space-time symmetries and derive the corresponding conservation laws.

Figure 1: *Emmy Noether (1882 - 1935)*

Homogeneity of time: "*The Lagrangian of a closed system cannot depend explicitly on time*". In practice this symmetry dictates that the result of an experiment does not depend upon the time that the experiment was performed. Due to the fact that the limits of the action integral depend upon time it is more convenient here to consider time variations of the Lagrangian. The total time-derivative of the Lagrangian is given by

$$\frac{dL}{dt} = \sum_i \left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right] + \frac{\partial L}{\partial t} \tag{1}$$

but since the Lagrangian does not depend explicitly on time then we have that

$$\frac{\partial L}{\partial t} = 0 \tag{2}$$

From (1) and (2) we have that

$$\frac{dL}{dt} = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] = \frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] \Rightarrow$$

$$\frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right] = 0$$

The quantity $H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$ is called the Hamiltonian and represents the energy of the system. Hence, t**he energy of the system is conserved as a consequence of homogeneity of time.**

Homogeneity of space: "*The Lagrangian should be invariant with respect to translations in space*". Physical processes should not depend on the space point that they take place. Consider a translation in space such as

$$\vec{r} \rightarrow \vec{r}\,' = \vec{r} + \vec{\varepsilon} \Rightarrow \quad \dot{\vec{r}}\,' = \dot{\vec{r}} \text{ and } \delta\vec{r} = \vec{\varepsilon} \quad ; \; \delta\dot{\vec{r}} = 0$$

Therefore, in this case only the generalized coordinates vary and not the generalized velocities. Hence,

$$\delta L = \sum_\alpha \frac{\partial L}{\partial \vec{r}_a}\delta\vec{r}_a = \vec{\varepsilon}\cdot\sum_\alpha \frac{\partial L}{\partial \vec{r}_a} = 0 \quad \Rightarrow$$

$$\delta L = \vec{\varepsilon}\cdot\sum_i \frac{\partial L}{\partial \vec{r}_a} = \vec{\varepsilon}\cdot\sum_\alpha \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_a} = 0 \quad \Rightarrow$$

$$\delta L = \vec{\varepsilon}\cdot\frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_a}\right] = 0 \quad \Rightarrow$$

$$\delta L = \vec{\epsilon}\cdot\frac{d\vec{p}}{dt} = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0$$

Therefore the homogeneity of space is the reason that **momentum is conserved as a consequence of homogeneity of space**.

Isotropy of space: "*The Mechanical properties of a system do not vary if it is rotated as a whole*". Lets consider a rotation of the system about an axis $\delta\hat{\phi} = \delta\phi\ \hat{k}$ where $\hat{k}$ is the unit vector along the z-axis. The variation of a vector **r** in this system due to rotation is equal to

$$\delta\vec{r} = \delta\hat{\varphi}\times\vec{r}$$

By taking derivative with respect of time we can calculate also the velocity variation

$$\delta\vec{v} = \delta\hat{\varphi}\times\vec{v}$$

The variation of the lagrangian due to this rotation is

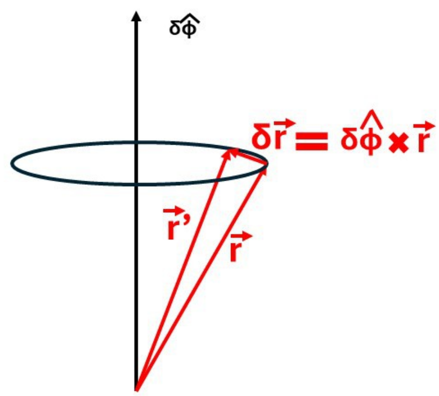

Figure 2: *Rotation of a vector*

$$\delta L = \sum_a\left[\frac{\partial L}{\partial\vec{r}_a}\delta\vec{r}_a + \frac{\partial L}{\partial\vec{v}_a}\delta\vec{v}_a\right] \Rightarrow$$

$$\delta L = \sum_a\left[\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{v}_a}\right)\delta\vec{r}_a + \frac{\partial L}{\partial\vec{v}_a}\delta\vec{v}_a\right] \Rightarrow$$

$$\delta L = \sum_a\left[\frac{d}{dt}\left[\frac{\partial L}{\partial\vec{v}_a}\right]\cdot\delta\hat{\varphi}\times\vec{r}_a + \frac{\partial L}{\partial\vec{v}_a}\cdot\delta\hat{\varphi}\times\vec{v}_a\right] \Rightarrow$$

$$\delta L = \sum_a\left[\frac{d\vec{p}_a}{dt}\cdot\delta\hat{\phi}\times\vec{r}_a + \vec{p}_a\cdot\delta\hat{\varphi}\times\vec{v}_a\right] \Rightarrow$$

$$\delta L = \sum_a\left[\delta\hat{\varphi}\cdot\vec{r}_a\times\frac{d\vec{p}_a}{dt} + \delta\hat{\varphi}\cdot\vec{v}_a\times\vec{p}_a\right] \Rightarrow$$

$$\delta L = \delta\hat{\varphi}\cdot\sum_a\left[\vec{r}_a\times\frac{d\vec{p}_a}{dt} + \vec{v}_a\times\vec{p}_a\right] \Rightarrow$$

$$\delta L = \delta\hat{\varphi}\cdot\frac{d}{dt}\sum_a\vec{r}_a\times\vec{p}_a \quad\Rightarrow\quad \delta L = \delta\hat{\varphi}\cdot\frac{d\vec{M}}{dt}$$

where

$$\vec{M} = \sum_a\vec{r}_a\times\vec{p}_a$$

is the angular momentum of the system.

By requiring that the physical processes and results do not change under rotation

$$\delta L = 0 \quad \Rightarrow \quad \frac{d\vec{M}}{dt} = 0$$

In conclusion, isotropy of space has as result that the **angular momentum is conserved.**

Action and Lagrangian in Quantum Field Theory

In Quantum Field Theory (QFT) fields are operators which can create or annihilate particles. In other words for each type of particle we have the corresponding field. We have the scalar fields which describes particles with spin-0 and satisfy the Klein-Gordon equation, we have the spinor fields which describe spin-1/2 particles and satisfy the Dirac equation and we have the vector fields which describe spin-1 particles. A QFT can also be formulated in terms of an action derived from a Lagrangian.

$$S = \int_\Omega dx^0 \mathcal{L} = \int dx^0 \int d^3x\, L = \int_\Omega d^4x\, L$$

where $\mathcal{L}$ is a Lagrangian and L is the Lagrangian density. Both are functions of the fields and their first derivatives. Ω is the integration 4 dimensional volume. The action can be written as[1]

$$S = \int_\Omega d^4x\, L(\Phi(x), \partial_\mu \Phi(x))$$

The action has the same units as $\hbar$. Hence, in the unit system used here, where $\hbar = c = 1$, the action is just a number. Given that d^4x has units of length in the fourth power this means that the Lagrangian density $L(\Phi(x), \partial_\mu \Phi(x))$ must have units of mass to the fourth power.

The Euler-Lagrange equations can also be derived from the action in a similar way as in Classical Physics. One considers a functional variation of the field $\delta\Phi$ which should vanish at the boundary $\partial\Omega$ of the 4 dimensional space of integration Ω and extract this way the general equations of motion.

$$\delta S = 0 \Rightarrow \int_\Omega d^4x \left[\frac{\partial L}{\partial \Phi} \delta\Phi + \frac{\partial L}{\partial(\partial_\mu \Phi)} \delta(\partial_\mu \Phi) \right] = 0 \Rightarrow$$

$$\int_\Omega d^4x \left[\frac{\partial L}{\partial \Phi} \delta\Phi + \frac{\partial L}{\partial(\partial_\mu \Phi)} \partial_\mu(\delta\Phi) \right] = 0 \Rightarrow$$

1 We suppressed here indices referring to different fields and their derivatives.

$$\int_{\Omega} d^4x \left[\frac{\partial L}{\partial \Phi} \delta\Phi \;+\; \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \delta\Phi \right) \;-\; \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \right) \delta\Phi \right] \;=\; 0$$

The second term can be written using Gauss's theorem as

$$\int_{\Omega} d^4x \left[\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \delta\Phi \right) \right] \;=\; \int_{\partial\Omega} d\,\Sigma^\mu \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \delta\Phi \right)$$

However, because $\delta\Phi$ vanishes on the boundary of the integration space $\partial\Omega$ the second term vanishes. Hence, we have that

$$\int_{\Omega} d^4x \left[\frac{\partial L}{\partial \Phi} \;-\; \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \right) \right] \delta\Phi \;=\; 0$$

and this gives us the Euler-Lagrange equations for the Field Φ.

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \right) \;-\; \frac{\partial L}{\partial \Phi} \;=\; 0$$

Similarly to Classical Mechanics the Lagrangian density is not unique because one can add to it a total derivative such as $\partial^\mu F_\mu(\Phi, x)$ and this will not change the equations of motion because the variations of the field $\delta\Phi$ vanish at the boundary.

Example 1: Consider the Lagrangian density for a real scalar field which is free of interactions.

$$L \;=\; \partial^\alpha \Phi(x) \partial_\alpha \Phi(x) - m^2 \Phi^2(x)$$

where m is the mass of the scalar particle represented by $\Phi(x)$.
(a) What is the unit of the field $\Phi(x)$?
(b) Derive the equation of motion for the field $\Phi(x)$.

Given that the Lagrangian density must have units of mass to the fourth power and that the derivatives have units of inverse length (same as units of mass) we conclude that the field $\Phi(x)$ must have units of mass.

$$\frac{\partial L}{\partial(\partial_\mu \Phi)} \;=\; \frac{\partial}{\partial(\partial_\mu \Phi)} \left(\partial^\alpha \Phi(x) \partial_\alpha \Phi(x) \right) \;=\; g^{\mu\alpha} \partial_\alpha \Phi(x) + \delta^\mu{}_\alpha \partial^\alpha \Phi(x) \quad \Rightarrow$$

$$\frac{\partial L}{\partial(\partial_\mu \Phi)} = 2\partial^\mu \Phi(x) \qquad (1)$$

$$\frac{\partial L}{\partial \Phi} = 2m^2 \Phi(x) \qquad (2)$$

From (1) and (2) and the Euler Lagrange equations we get

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \right) - \frac{\partial L}{\partial \Phi} = 0 \Rightarrow \left(\partial_\mu \partial^\mu + m^2 \right) \Phi(x) = 0$$

This is the Klein-Gordon equation. In conclusion **this Lagrangian density has a single digree of freedom (field) and describes a scalar or a pseudoscalar field with mass *m*.**

Example 2: Consider the Lagrangian density for a complex scalar field which is free of interactions

$$L = \partial^\alpha \Phi^*(x) \partial_\alpha \Phi(x) - m^2 \Phi^*(x) \Phi(x)$$

where m is the mass of the scalar particles represented by $\Phi(x)$ and $\Phi^*(x)$
(a) What is the unit of the fields $\Phi(x)$ and $\Phi^*(x)$?
(b) Derive the equation of motion for the fields $\Phi(x)$ and $\Phi^*(x)$.

The fields $\Phi(x)$ and $\Phi^*(x)$ have units of mass.

$$\frac{\partial L}{\partial(\partial_\mu \Phi^*)} = \frac{\partial}{\partial(\partial_\mu \Phi^*)} \left(\partial^\alpha \Phi^* \partial_\alpha \Phi(x) \right) = g^{\mu\alpha} \partial_\alpha \Phi(x) = \partial^\mu \Phi(x) \qquad (1)$$

$$\frac{\partial L}{\partial \Phi^*} = m^2 \Phi(x) \qquad (2)$$

From (1) and (2) and the Euler Lagrange equations we get

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi^*)} \right) - \frac{\partial L}{\partial \Phi^*} = 0 \Rightarrow \left(\partial_\mu \partial^\mu + m^2 \right) \Phi(x) = 0$$

Similarly by differentiating with respect to Φ we get the equations of motion for the field Φ^* .

$$\left(\partial_\mu \partial^\mu + m^2\right)\Phi^*(x) = 0$$

So this Lagrangian density describes two scalar particles which have the same mass. Once we introduce an electromagnetic field we will observe that these are two oppositely charged scalars which have the same mass.

In both examples (1) and (2) we see that the Lagrangian density has two distinct parts. A kinetic part which is a function of the field derivatives to the second power and the mass part which is a function of the field to the second power.

Example 3: The Vector Field Lagrangian density is given by

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

Where $A^\mu = (\Phi ; \vec{A})$ is the vector potential and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ the Maxwell tensor which satisfies $F^{\mu\nu} = -F^{\nu\mu}$.

(a) What is the unit of the field A^μ ?
(b) Derive the equations of motion for the field A^μ .

The field A^μ has units of mass and has 4 components. So there are four degrees of freedom. There is no mass term here becuase this Lagrangian density will describe photons which are massless.

The Lagrangian density can be written as

$$L = -\frac{1}{4}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) F^{\alpha\beta} = -\frac{1}{2} \partial_\alpha A_\beta F^{\alpha\beta}$$

Using this Lagrangian density and the Euler-Lagrange equations we can extract the equations of motion for A^μ .

$$\frac{\partial L}{\partial \partial_\mu A_\nu} = -\frac{1}{2} \frac{\partial}{\partial \partial_\mu A_\nu}\left(\partial_\alpha A_\beta F^{\alpha\beta}\right) = -\frac{1}{2} \frac{\partial}{\partial \partial_\mu A_\nu}\left(\partial_\alpha A_\beta \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\right) \Rightarrow$$

$$\frac{\partial L}{\partial \partial_\mu A_\nu} = -\frac{1}{2}\left[\delta^\mu{}_\alpha \delta^\nu{}_\beta \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\right] - \frac{1}{2}\left[\partial_\alpha A_\beta \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}\right)\right] \Rightarrow$$

$$\frac{\partial L}{\partial \partial_\mu A_\nu} = -\frac{1}{2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = F^{\mu\nu} \quad (1)$$

$$\frac{\partial L}{\partial A_\nu} = 0 \quad (2)$$

Finally substituting (1) and (2) into the Euler Lagrange equations

$$\partial_\mu\left(\frac{\partial L}{\partial \partial_\mu A_\nu}\right) - \frac{\partial L}{\partial A_\nu} = 0$$

we get two of the Maxwell equations in the absence of the electromagnetic current and charge density.

$$\partial_\mu F^{\mu\nu} = 0$$

The other two Maxwell equations come from the dual tensor

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

where

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

We would like to identify the photon with the field A^μ . However, a physical photon is massless and has only two degrees of freedom (Left Handed/Right Handed) whilst A^μ has four degrees of freedom. Gauge invariance of the Maxwell equations and the freedom to choose a specific gauge permits us to solve Maxwell's equations and through this process only two of the four degrees of freedom remain which correspond to the two different kinds of photon polarization. However, the details of this are beyond this course and can be found in text books of QFT[1].

In the presence of charge and/or current density the Lagrangian density acquires an extra term where the 4-vector current couples to the 4-vector potential

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{4\pi}{c}J_\mu A^\mu$$

In this case the derivative with respect to the field is no longer zero.

1 For quantizing the photon field in a non-manifestly covariant way using the Coulomb Gauge look in Bjorken Drell, Quantum Fields, Vol. II. For quantizing the photon field in a manifestly covariant way via the Gupta-Bleuler method look in Itzykson and Zuber, Quantum Field Theory.

Instead we have that

$$\frac{\partial L}{\partial A_\nu} = \frac{4\pi}{c} J^\nu$$

Which will give us the Maxell equations at the presence of charge and/or current density.

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

Example 4: Consider the Lagrangian density of the Dirac Field.

$$L = \frac{i}{2}\left(\bar{\Psi}\gamma^\mu \partial_\mu \Psi - (\partial_\mu \bar{\Psi})\gamma^\mu \Psi\right) - m\bar{\Psi}\Psi$$

(a) What is the unit of the field Ψ ?
(b) Derive the equations of motion for Ψ.
(c) Show that the Lagrangian density

$$L = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi$$

is equivalent and describes also a fee spin-½ fermion.

The field Ψ has units of mass to the power of 3/2 because the derivative has units of mass and this way the Lagrangian density has units of mass to the fourth power. In general spin ½ fermions have mass so this Lagrangian density has a kinetic part which depends upon the field derivatives and a mass part which depends upon the square of the field.

Here we will extract the equations of motion for the field Ψ. Students are encouraged to do the same for the field $\bar{\Psi}$. We start by calculating the derivatives of the Lagrangian density with respect to the field and its derivative.

$$\frac{\partial L}{\partial \partial_\mu \bar{\Psi}} = -\frac{i}{2}(\gamma^\mu)_{ab}\Psi_b \quad , \quad \frac{\partial L}{\partial \bar{\Psi}_a} = +\frac{i}{2}(\gamma^\mu)_{ab}\partial_\mu \Psi_b - m\Psi_a$$

Next we substitute these in to the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial L}{\partial \partial_\mu \bar{\Psi}_a}\right) - \frac{\partial L}{\partial \bar{\Psi}_a} = 0 \Rightarrow$$

to get

$$-\frac{i}{2}(\gamma^\mu)_{ab}\partial_\mu \Psi_b - \frac{i}{2}(\gamma^\mu)_{ab}\partial_\mu \Psi_b + m\Psi_a = 0 \Rightarrow$$

$$-i(\gamma^\mu)_{ab}\partial_\mu \Psi_b + m\Psi_a = 0 \Rightarrow$$

$$[i\gamma^\mu \partial_\mu - m]\Psi(x) = 0$$

Which is the Dirac equation.

However, we are free to add to the Dirac Lagrangian density a total derivative term such as

$$\frac{i}{2}\partial_\mu\left(\bar{\Psi}\gamma^\mu \Psi\right)$$

which will not change the equations of motion since the variations of the field vanish at the boundary. Therefore we have that

$$L = \frac{i}{2}\left(\bar{\Psi}\gamma^\mu \partial_\mu \Psi - (\partial_\mu \bar{\Psi})\gamma^\mu \Psi\right) - m\bar{\Psi}\Psi + \frac{i}{2}\partial_\mu\left(\bar{\Psi}\gamma^\mu \Psi\right) \Rightarrow$$

$$L = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi$$

Global and Local Symmetries of the Lagrangian Density

Consider a Lagrangian density L which is a function of a field $\Phi(x)$ and its derivative $\partial_\mu \Phi(x)$ and assume that the field $\Phi(x)$ is a solution of the Euler Lagrange equations[1]. Suppose next that the Lagrangian density is invariant under the transformation

$$\Phi(x) \to \Phi(x) + i e \alpha \Phi(x) \Rightarrow \delta\Phi(x) = i e \alpha \Phi(x) \qquad (1)$$

where **e** is a constant and α could be a constant or it could be a function of x. As seen here this transformation is very different than the space-time transformations we considered before. This is a transformation which leaves the space-time point the same but changes the field in a way that it is proportional to the field and if often called *internal transformation*. Furthermore, the variation of the field is non-zero everywhere in space time. Since this is a continuous transformation we expect from Noether's theorem that it should result to a conserved quantity.

1 For simplicity we assume only one field but this can easily be generalized to any number of fields.

Indeed suppose that the action remains invariant under this transformation. Then we have that

$$\int_{\Omega} d^4x \left[\frac{\partial L}{\partial \Phi} \delta \Phi \ + \ \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \delta \Phi \right) \ - \ \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \right) \delta \Phi \right] \ = \ 0 \ \Rightarrow$$

However, since the Euler-Lagrange equations are satisfied by $\Phi(x)$, the first and the third term cancel out. Furthermore, since $\delta\Phi(x) \ = \ i\, e\, a\, \Phi(x)$ does not vanish the second term survives and finally we have that.

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \Phi)} \delta \Phi \right) \ = \ 0$$

This can be generalized in the case we have a number of degees of freedom (a number of fields) as

$$\partial_\mu \left(\sum_a \frac{\partial L}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a \right) \ = \ 0$$

Next we need to look more closely on the field variation shown in (1). If the quantity **α** is just a constant then then transformation is called **global transformation** and the corresponding symmetry is called **global symmetry.**

In this case there is a conserved current

$$J^\mu \ = \ \sum_a \frac{\partial L}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a \quad \text{such that} \quad \partial_\mu J^\mu \ = \ 0 \qquad (2)$$

In conclusion, **the global symmetry of the Lagrangian density resulted to a conserved current**. The continuity equation shown in (2) can be written as

$$\frac{\partial J^\mu}{\partial x^\mu} \ = \ 0 \ \Rightarrow \ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \ = \ 0 \ \Rightarrow \ \frac{dQ}{dt} \ = \ -\int_S \vec{J} \cdot d\vec{s}$$

which gives the **conservation of charge**. **Hence, internal symmetries are responsible for the conservation of charges of the different interactions.**

Example 1: Consider the complex scalar field Lagrangian density

$$L = \partial^{\alpha}\Phi^{*}(x)\partial_{\alpha}\Phi(x) - m^{2}\Phi^{*}(x)\Phi(x)$$

which is invariant under the global phase transformation

$$\Phi(x) \rightarrow e^{ie}\Phi(x) \Rightarrow \delta\Phi \approx ie\Phi(x)$$

where **e** is a constant. Based on what we have presented there is a conserved current

$$J^{\mu} = \sum_{a}\frac{\partial L}{\partial(\partial_{\mu}\Phi_{a})}\Phi_{a} = \frac{\partial L}{\partial(\partial_{\mu}\Phi)}(ie\Phi) + \frac{\partial L}{\partial(\partial_{\mu}\Phi^{*})}(-ie\Phi^{*}) \Rightarrow$$

$$J^{\mu} = \partial^{\mu}\Phi^{*}(+ie\Phi) + \partial^{\mu}\Phi(-ie\Phi^{*}) \Rightarrow$$

$$J^{\mu} = ie\left(\Phi\partial^{\mu}\Phi^{*} - \Phi^{*}\partial^{\mu}\Phi\right)$$

Which is the conserved current which we derived using the Klein-Gordon equation and in fact it is trivial to show that this is conserved

$$\partial_{\mu}J^{\mu} = 0$$

using the Klein-Gordon equation. So, this tells us that charge is conserved in a world made of opposite charged scalar or pseudoscalar particles.

Example 2: Consider the Dirac Lagrangian density

$$L = \frac{i}{2}\left(\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - (\partial_{\mu}\bar{\Psi})\gamma^{\mu}\Psi\right) - m\bar{\Psi}\Psi$$

which is invariant under the global phase transformation where

$$\Psi(x) \rightarrow e^{-ie}\Psi(x) \Rightarrow \delta\Psi \approx -ie\Psi(x)$$

and

$$\bar{\Psi}(x) \rightarrow e^{+ie}\bar{\Psi}(x) \Rightarrow \delta\Psi \approx +ie\bar{\Psi}(x)$$

based on this the conserved current is

$$J^{\mu} = \frac{\partial L}{\partial(\partial_{\mu}\Psi)}(-ie\Psi) + (+ie\bar{\Psi})\frac{\partial L}{\partial(\partial_{\mu}\bar{\Psi})} \Rightarrow$$

$$J^{\mu} = \frac{i}{2}\bar{\Psi}\gamma^{\mu}(-ie\Psi) - (+ie\bar{\Psi})\frac{i}{2}\gamma^{\mu}\Psi \Rightarrow$$

$$J^{\mu} = e\bar{\Psi}\gamma^{\mu}\Psi$$

which is the Dirac current we extracted when we studied the Dirac equation and one can show explicitly that is conserved using the Dirac equation.

So far we have considered internal transformations which represented a rotation at a constant angle/phase on the complex plane. In other words we considered that the same phase change occurs at every space-time point. Next we will consider rotations on the complex plane where the phase varies in space-time. These are called ***local transformations*** or better known as ***gauge transformations.*** Lets consider the simplest gauge transformation

$$\Phi(x) \to \Phi'(x) = e^{iea(x)}\Phi(x) \Rightarrow \delta\Phi \approx iea(x)\Phi(x) \qquad \text{(G)}$$

and lets study how the Lagrangian density of the complex scalar field transforms under this transformation.

Substituting $\Phi(x) \to \Phi'(x)$ in to

$$L = \partial^{\mu}\Phi^{*}(x)\partial_{\mu}\Phi(x) - m^{2}\Phi^{*}(x)\Phi(x)$$

we find that the Lagrangian density is not invariant under the gauge transformation G. Whilst the mass term is invariant, the derivatives result to extra terms

$$\partial_{\mu}\Phi'(x) = e^{iea(x)}\partial_{\mu}\Phi(x) + ie\partial_{\mu}a(x)e^{iea(x)}\Phi(x)$$

which depend upon $\partial_{\mu}a(x)$. There is no reason why physical phenomena should not be invariant when the phases of the fields which describe the particles depend on space-time. Therefore some other physical process has not been taken in to account which would cancel the extra terms that make the Lagrangian density non-invariant. The answer to this is that we have not taken in to account that our charged particles in order to interact need to be able to emit and absorb photons.

So we introduce photons in to this theory and we require that the resulting Lagrangian density is invariant under the gauge transformation G.

This is accomplished by introducing the **covariant derivative** which replaces the derivative in the Lagrangian density

$$D_\mu = \partial_\mu - i e A_\mu$$

where A_μ is a vector field which we will identify later with the photon.

The new Lagrangian density is

$$L = (D^\mu \Phi)^*(x) D_\mu \Phi(x) - m^2 \Phi^*(x)\Phi(x)$$

The field A_μ should also transform under the gauge transformation in such a way so that it cancels the unwanted terms which depend on $\partial_\mu \alpha(x)$. So under the gauge transformation we have that

$$\Phi(x) \rightarrow \Phi'(x) = e^{i e \alpha(x)} \Phi(x)$$

and

$$A_\mu \rightarrow A'_\mu = ?$$

and the question now is if there exists a transformation of the vector field which leaves the new Lagrangian density invariant.

The modified Lagrangian density can be made invariant under the Gauge transformation if we require that

$$\left(\partial_\mu - i e A_\mu'\right)\Phi'(x) = e^{i e \alpha(x)}\left(\partial_\mu - i e A_\mu\right)\Phi(x) \qquad (1)$$

where the primed fields are the fields which have been transformed by the gauge transformation.

$$\left(\partial_\mu - i e A_\mu'\right)\Phi'(x) = \left(\partial_\mu - i e A_\mu'\right)\left[e^{i e \alpha(x)}\Phi(x)\right] =$$

$$e^{i e \alpha(x)}\partial_\mu \Phi(x) + i e \partial_\mu \alpha(x) e^{i e \alpha(x)}\Phi(x) - i e A_\mu' e^{i e \alpha(x)}\Phi(x) \qquad (2)$$

From (1) and (2) we have then that

$$e^{i\,e\,\alpha(x)}\partial_\mu \Phi(x) \;+\; i\,e\,\partial_\mu \alpha(x) e^{i\,e\,\alpha(x)} \Phi(x) - i\,e\,A_\mu{}' e^{i\,e\,\alpha(x)} \Phi(x) \;=$$

$$e^{i\,e\,\alpha(x)}\partial_\mu \Phi(x) - i\,e\,A_\mu e^{i\,e\,\alpha(x)} \Phi(x) \quad \Rightarrow$$

$$\partial_\mu \alpha(x) - A_\mu{}' \;=\; -A_\mu \quad \Rightarrow \quad A_\mu{}' \;=\; A_\mu \;+\; \partial_\mu \alpha(x)$$

Hence, the Lagrangian

$$L \;=\; (D^\mu \Phi)^*(x) D_\mu \Phi(x) - m^2 \Phi^*(x) \Phi(x)$$

is invariant under the gauge transformation where

$$\Phi(x) \to \Phi'(x) \;=\; e^{i\,e\,\alpha(x)} \Phi(x)$$

$$A_\mu \;\to\; A_\mu{}' \;=\; A_\mu \;+\; \partial_\mu \alpha(x)$$

but we are not done yet because the field A_μ which enters now in the Lagrangian density has no kinetic term. We would like that A_μ satisfies Maxwell's equation so we need to add the Maxwell Lagrangian density

$$L_{Maxwell} \;=\; -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad \text{where} \quad F^{\mu\nu} \;=\; \partial^\mu A^\nu - \partial^\nu A^\mu$$

We see that $F^{\mu\nu} \;=\; \partial^\mu A^\nu - \partial^\nu A^\mu$ is gauge invariant. Hence, the final Lagrangian which is gauge invariant will be

$$L \;=\; (D^\mu \Phi)^*(x) D_\mu \Phi(x) - m^2 \Phi^*(x) \Phi(x) \;-\; \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

In the rest we will try to understand the physical meaning and the predictions of this Lagrangian density. Another way of writing this Lagrangian density is

$$L \;=\; \partial^\mu \Phi^* \partial_\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\,e\left(\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*\right) A_\mu + e^2 \Phi^* \Phi\, A_\mu A^\mu$$

We will describe next what is the physical meaning of the different terms of the Lagrangian density.

The first two terms describe two charged scalar or pseudoscalar particles with equal mass $\boldsymbol{m}$. The third term gives the free Maxwell's equations for the field A_μ. In other words it tells us that the vector field we introduced in the covariant derivative is indeed the photon.

The last two terms describe how the scalar/pseudoscalar particles interact with the photons. The fourth term describes the coupling of the photon with the well known Klein-Gordon conserved current. As we see there we have two Φ fields (incoming and outgoing) and one A_μ field involved. Therefore it describes a charged particle Φ either absorbing or emitting a photon. The fifth term involves two Φ fields and two A_μ fields and therefore it describes the absorption or emission of two photons at the same space time point by a Φ particle. The fourth and the fifth terms are the interaction terms and contribute as sources to Maxwell's equations. These two terms can be represented diagrammatically as shown in Figure 3.

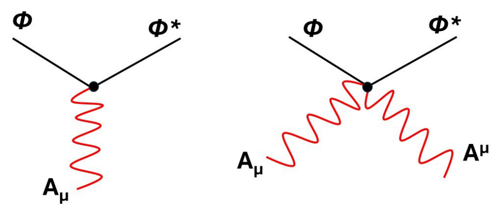

Figure 3: *Interactions of a scalar or pseudoscalar particle with a photon (left) and two photons (right)*

The theory that we just worked out is called **Scalar Quantum Electrodynamics** and describes electromagnetic interactions of charged scalar or pseudoscalar particles.

As demonstrated here the demand for **gauge invariance resulted in predicting the correct interaction between charged particles and photons** and therefore justified the introduction of the covariant derivative which in the past we introduced somewhat arbitrarily via the minimal coupling. In other words gauge invariance is the origin of the source terms in Maxwell's equations. Predicting the correct couplings between matter particles and gauge bosons is one of the features that made gauge theories attractive.