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Time Reversal

Time Reversal is the operation according which the space-time coordinates change as follows:

$$(\vec{x},t)\rightarrow(\vec{x}',t') = (\vec{x},-t)$$

Similarly to parity and charge conjugation, theories whose equations are invariant under time reversal are said to 'have the Time Reversal symmetry'. One wonders what does this exactly mean in terms of the physical processes. Suppose that an interaction causes the transition from a state A to a state B $(A\rightarrow B)$. If this interaction has the Time Reversal symmetry then the transition from B to A $(B\rightarrow A)$ can also occur with the same transition amplitude as the transition $A\rightarrow B$. In other words by observing and measuring any of the two transitions one cannot draw conclusions about the direction of time. **Therefore, no experiment using this process can be designed which can detect the direction of time.**

Having introduced Time Reversal we can proceed to study the invariance of the Dirac equation under Time Reversal.

The Dirac Equation under Time Reversal

We will consider the Dirac equation in the case where an electron or a positron is coupled to an electromagnetic field which is described by scalar potential $\Phi(\vec{x},t)$ and vector potential $\vec{A}(\vec{x},t)$ i.e $A^{\mu}(x) = (\Phi(\vec{x},t),\vec{A}(\vec{x},t))$.

$$\left[i\gamma^{\mu}(\partial_{\mu}+ie\,A_{\mu}(\vec{x},t))-m\right]\Psi(x) = 0 \qquad e<0$$

To study how this equation transforms under time reversal we need to write it in a non-covariant form because the transformation under consideration involves only time.

$$i\frac{\partial\Psi(\vec{x},t)}{\partial t} = \left[\vec{\alpha}\cdot(-i\vec{\nabla}-e\vec{A}(\vec{x},t))+m\,\beta+e\Phi\right]\Psi(\vec{x},t) = H_D\Psi(\vec{x},t) \quad (1)$$

As seen here in the Dirac Hamiltonian we have replaced the space-time derivatives ∂_{μ} with the covariant derivate $D_{\mu} = \partial_{\mu}+i\,e\,A_{\mu}$. This substitution, which is usually called *minimal coupling*, will be justified in the next chapter when we discuss about local gauge invariance.

Next we will study the conditions under which equation (1) is invariant under time reversal.

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Lets define the time reversal operator acting on spinors as

$$\Psi'(\vec{x},t') = G\Psi(\vec{x},t) \quad \text{where} \quad t' = -t \tag{2}$$

From (1) and (2) we have that

$$G i G^{-1}\frac{\partial \Psi'(\vec{x},t')}{\partial t'} = -G H_D G^{-1}\Psi'(\vec{x},t')$$

If the Dirac equation is to remain invariant then it we should have either that

$$GiG^{-1} = -i \quad \text{and} \quad G H_D G^{-1} = H_D$$

or

$$GiG^{-1} = i \quad \text{and} \quad G H_D G^{-1} = -H_D$$

We can get a hint as to which one of the two choices should we follow by taking in to account the expected change of the vector and scalar potentials under Time Reversal

$$\vec{A}(\vec{x},t) = -\vec{A}'(\vec{x},-t) = -\vec{A}'(\vec{x},t') \quad \text{and} \quad \Phi(\vec{x},t) = +\Phi'(\vec{x},-t)$$

This expectation is justified because the vector potential is ultimately related to the electric current which changes sign under Timer Reversal so we expect it to also change sign whist the scalar potential is proportional to the charge which does not change under Time Reversal.

Under Time Reversal $\vec{\nabla}=\vec{\nabla}'$. However, we wish that the term with the $\vec{\nabla}$ transforms like the vector potential (with a minus). So there must be a complex conjugation operation involved in this transformation. Therefore it seems that we need to choose the first option where $GiG^{-1} = -i$.

So lets try

$$\Psi'(t')=T\Psi^*(t) \Rightarrow \Psi(t)=(T^{-1}\Psi'(t'))^* \tag{3}$$

where T is a 4x4 matrix to be determined. For the rest of this study we suppress the space arguments because they are not relevant here.

From (1) and (3) we have then that

$$i\frac{\partial}{\partial t}\left[T^{-1}\Psi'(t')\right]^* = \left[\vec{\alpha}(-i\vec{\nabla}-e\vec{A}(t))+e\Phi+m\beta\right]\left[T^{-1}\Psi'(t')\right]^* \Rightarrow$$

$$-i\frac{\partial}{\partial t}\left[T^{-1}\Psi'(t')\right] = \left[\vec{\alpha}^*(+i\vec{\nabla}-e\vec{A}(t))+e\Phi+m\beta^*\right]\left[T^{-1}\Psi'(t')\right] \Rightarrow$$

$$+i\frac{\partial}{\partial t'}\left[T^{-1}\Psi'(t')\right] = \left[\vec{\alpha}^*(+i\vec{\nabla}+e\vec{A}'(t'))+e\Phi+m\beta^*\right]\left[T^{-1}\Psi'(t')\right] \Rightarrow$$

$$+i\frac{\partial}{\partial t'}\Psi'(t') = \left[(T\vec{\alpha}^*T^{-1})(+i\vec{\nabla}+e\vec{A}'(t'))+e\Phi+m\,T\beta^*T^{-1}\right]\Psi'(t') \Rightarrow$$

$$+i\frac{\partial}{\partial t'}\Psi'(t') = \left[-(T\vec{\alpha}^*T^{-1})(-i\vec{\nabla}-e\vec{A}'(t'))+e\Phi+m\,T\beta^*T^{-1}\right]\Psi'(t')$$

So we want to find a matrix T such that

$$T\,i\,T^{-1} = i \quad , \quad -T\vec{\alpha}^*T^{-1} = \vec{\alpha} \quad \text{and} \quad T\beta^*T^{-1} = \beta \qquad (4)$$

Recall that

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow (\alpha^1)^* = \alpha^1,\ (\alpha^3)^* = \alpha^3,\ (\alpha^2)^* = -\alpha^2, \beta = \beta^*$$

Then from (4) we have that

$$[T,\beta] = 0,\quad [T,\alpha^2] = 0,\quad \{T,\alpha^1\} = 0,\quad \{T,\alpha^3\} = 0 \quad (5)$$

The requirement $[T,\beta] = 0$ can be satisfied only if $T = a^i a^j \quad i,j = 1,2,3$ whilst the requirement $[T,\alpha^2] = 0$ can be satisfied if $T = a^1 a^3$. This satisfies also the anti-commutator relations in (5)

Hence, we choose

$$T = i\gamma^1\gamma^3$$

Therefore,

$$\Psi'(\vec{x},t') = i\gamma^1\gamma^3\Psi^*(\vec{x},t)$$

Exercise: Show that

$$[\gamma^0,\gamma^1\gamma^3] = 0,\ [\alpha^2,\gamma^1\gamma^3] = 0,\ \{\alpha^1,\gamma^1\gamma^3\} = 0,\ \{\alpha^3,\gamma^1\gamma^3\} = 0$$

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Exercise: The spinor

$$v_L = \sqrt{E}\begin{pmatrix}1\\ \vec{\sigma}\cdot\hat{p}\end{pmatrix}\chi^{(2)}e^{-ipx}$$

describes a neutrino which is taken to be a massless spin ½ fermion with negative helicity moving in the positive $\hat{p}$ direction. $\hat{p}$ is the unit 3-vector at direction of the momentum, $\vec{\sigma}$ are the Pauli matrices, $p,\ x$ are the 4-momentum and 4-position and $\chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$. Apply Time Reversal on this spinor and give a physical interpretation for your result.

Solution:

$$(v_L)^T = i\gamma^1\gamma^3\left(v_L(-t)\right)^* = \begin{pmatrix}-\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix}\sqrt{E}\begin{pmatrix}1\\ (\vec{\sigma})^*\cdot\hat{p}\end{pmatrix}\chi^{(2)}e^{i\left(p^0(-x^0)-\vec{p}\cdot\vec{x}\right)} \Rightarrow$$

$$(v_L)^T = \sqrt{E}\begin{pmatrix}1\\ \vec{\sigma}\cdot(-\hat{p})\end{pmatrix}\chi^{(1)}e^{-i\left(p^0x^0-(-\vec{p}\cdot\vec{x})\right)}$$

We observe that $(v_L)^T$ represents a massless spin ½ fermion which has momentum and spin opposite than v_L. So it is a negative helicity spin ½ fermion moving 'backwards'.

The result of this operation is shown pictorially in Fig. 1.

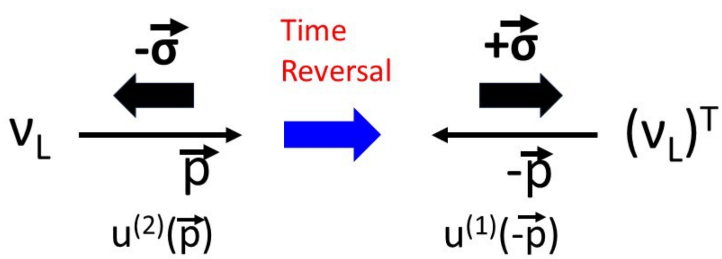

Figure 1: *The Time Reversal operation on a neutrino spinor.*

As seen here both $\vec{\sigma}$ and $\vec{p}$ have changed sign, so the result is still a negative helicity spinor. However, the new spinor moves in the opposite direction than the original.

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The CPT Theorem

Parity, Charge Conjugation and Time Reversal are called **discrete symmetries** because the corresponding quantities can take discrete values (plus or minus one) as opposed for example to translational or rotational symmetries which are continuous and can translate/rotate a system by any length/angle. Continuous symmetries correspond to additive quantum numbers (momentum, angular momentum etc). Discrete symmetries correspond to quantum numbers that are multiplicative as we have already seen with Parity and Charge Conjugation.

The CPT theorem states that all Quantum Field Theories must be invariant under the combined Charge Conjugation, Parity and Time Reversal transformations. This theorem can be proven for free field theories subject to requirements that are very fundamental in physics: Lorentz invariance, locality, and hermiticity. For interactive field theories it can only be proven when additional caveats are assumed so it is accepted as an axiom.

A CPT transformation transforms a particle of a given mass, energy and momentum to an antiparticle of the same mass, energy and momentum.

Indeed the stringiest experimental test of this theorem today comes from measurements of the difference of mass and width between the $\boldsymbol{K^0}$ and $\boldsymbol{\bar{K}^0}$ strange mesons.

$$2\frac{|m_{K^0}-m_{\bar{K}^0}|}{(m_{K^0}+m_{K^0})} < 6\times 10^{-19} \qquad 2\frac{|\Gamma_{K^0}-\Gamma_{\bar{K}^0}|}{(\Gamma_{K^0}+\Gamma_{K^0})} = (8\pm 8)\times 10^{-18}$$

The CPT theorem dictates that the properties of matter should be the same as those of anti-matter. For example the energy spectrum of Anti-Hydrogen atoms must be identical to that from Hydrogen atoms.

The Dirac Equation under CPT

In this section we will study the invariance of the Dirac equation under the combined transformation of Parity (P), Charge Conjugation (C), and Time Reversal (T) transformations. As we have seen the effect of a Parity, Charge Conjugation and Time Reversal transformations on a solution of the Dirac equation is

$$\begin{aligned} \Psi^P(\vec{x},t) &= \gamma^0\Psi(-\vec{x},t) \\ \Psi^C(\vec{x},t) &= i\gamma^2\Psi(\vec{x},t)^* \\ \Psi^T(\vec{x},t) &= i\gamma^1\gamma^3\Psi(\vec{x},-t)^* \end{aligned}$$

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Therefore the transformation under both P and C gives

$$\Psi^{CP}(\vec{x},t) = i\gamma^2\left[\gamma^0\Psi(-\vec{x},t)\right]^* = i\gamma^2\gamma^0\Psi^*(-\vec{x},t)$$

and by applying next the Time Reversal transformation we obtain

$$\Psi^{TCP}(\vec{x},t) = i\gamma^1\gamma^3\left[i\gamma^2\gamma^0\Psi^*(-\vec{x},-t)\right]^* = -\gamma^1\gamma^3\gamma^2\gamma^0\Psi(-\vec{x},-t) \Rightarrow$$

$$\Psi^{TCP}(\vec{x},t) = i\gamma^5\Psi(-\vec{x},-t) \quad \text{or simply} \quad \Psi^{TCP}(x) = i\gamma^5\Psi(-x,) \qquad (1)$$

It is easy to show that $\Psi^{TCP}(x)$ satisfies the Dirac equation. From (1) we have that

$$\Psi(x) = -i\gamma^5\Psi^{TCP}(-x) = -i\gamma^5\Psi^{TCP}(x') \qquad x' = -x$$

and substituting this into the Dirac equation one gets

$$\left(i\gamma^\mu\partial_\mu - m\right)\Psi(x) = 0 \Rightarrow \left(i\gamma^\mu\partial_\mu - m\right)(-i\gamma^5)\Psi^{TCP}(x') = 0 \Rightarrow$$

$$(-i\gamma^5)\left(-i\gamma^\mu\partial_\mu - m\right)\Psi^{TCP}(x') = 0 \Rightarrow \left(i\gamma^\mu\partial_\mu' - m\right)\Psi^{TCP}(x') = 0$$

Hence, the Dirac equation is invariant under the TCP transformation.

$$\Psi(x) \rightarrow \Psi^{TCP}(x) = i\gamma^5\Psi(-x)$$

Next we would like to see what kind of physical particle does $\Psi^{TCP}(x)$ describe. We will do this with the following exercise.

Exercise 1: TCP transform $\Psi^{(1)}, \Psi^{(2)}$ which are free positive energy solutions of the Dirac equation which describe spin ½ fermions with positive and negative helicity respectively.

Solution: We have that

$$\Psi^{(1,2)} = \sqrt{E+M}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+M} \end{pmatrix}\chi^{(1,2)}e^{-i\,px}$$

and using the TCP transformation from (1) we get

$$\left[\Psi^{(1)}\right]^{TCP} = i\gamma^5 \Psi^{(1)}(-x) = i\sqrt{E+M}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+M} \\ 1 \end{pmatrix}\chi^{(1)} e^{+ipx} = i\,v^{(2)}(\vec{p})e^{+ipx} \quad (2)$$

$$\left[\Psi^{(2)}\right]^{TCP} = i\gamma^5 \Psi^{(2)}(-x) = i\sqrt{E+M}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+M} \\ 1 \end{pmatrix}\chi^{(2)} e^{+ipx} = i\,v^{(1)}(\vec{p})e^{+ipx} \quad (3)$$

The results from (2) and (3) demonstrate that **this operation acting on a spin ½ fermion changed particle with antiparticle, while leaving mass, energy and momentum the same. However, the TCP transformation flips the helicity of the fermion**. This makes sense since the momentum will change sign twice, once due to parity and once due to time reversal, so it will not change sign under TCP, while the spin will change sign once during time reversal. Hence, helicity changes sign under TCP or CPT (the order we perform the three operations does not make a difference). In Table 1 we show how spin, momentum and helicity change under the three different discrete transformations.

	$\vec{\sigma}$	$\hat{p}$	$\vec{\sigma}\cdot\hat{p}$
P	$+\vec{\sigma}$	$-\hat{p}$	$-\vec{\sigma}\cdot\hat{p}$
C	$+\vec{\sigma}$	$+\hat{p}$	$+\vec{\sigma}\cdot\hat{p}$
T	$-\vec{\sigma}$	$-\hat{p}$	$+\vec{\sigma}\cdot\hat{p}$
CPT	$-\vec{\sigma}$	$+\hat{p}$	$-\vec{\sigma}\cdot\hat{p}$

Table 1: *The effect of discrete symmetries on Helicity.*

We see that Charge Conjugation and Time Reversal do not change helicity but Parity does. Note that in Table 1 the values for spin and momentum in the case of Charge Conjugation refer to the physical anti-particle and not to those of the negative energy solution which are opposite.

Maxwell's Equations under CPT

In this section we will study the invariance of the classical Maxwell equations under parity, charge conjugation and time reversal. The Maxwell equations in the CGS system of units are:

$$\vec{\nabla}\cdot\vec{E}(\vec{x},t) = 4\pi\rho(\vec{x},t) \quad (1) \qquad \vec{\nabla}\cdot\vec{B}(\vec{x},t) = 0 \quad (2)$$

$$\vec{\nabla}\times\vec{E}(\vec{x},t) = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad (3) \qquad \vec{\nabla}\times\vec{B}(\vec{x},t) = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \quad (4)$$

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As is well known, equation (2) implies that

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{5}$$

and equations (3) and (5) give

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{6}$$

Parity: First we consider parity transformations under which the coordinates of the reference frame change according $(\vec{x},t)\rightarrow(\vec{x}',t)$ where $\vec{x}' = -\vec{x}$. Due to this the charge and current densities as well as the fields change accordingly.

$$\rho(\vec{x},t)\rightarrow\rho'(\vec{x}',t) \quad (7) \qquad \vec{J}(\vec{x},t)\rightarrow\vec{J}'(\vec{x}',t) \quad (8)$$

$$\vec{E}(\vec{x},t)\rightarrow\vec{E}'(\vec{x}',t) \quad (9) \qquad \vec{B}(\vec{x},t)\rightarrow\vec{B}'(\vec{x}',t) \quad (10)$$

Next we need to investigate if there is such a transformation of the fields which leaves the Maxwell equations invariant under parity. Since $(\vec{x},t)$ and $(\vec{x}',t)$ refer to the same space-time point it is rather obvious that under parity we have that

$$\rho(\vec{x},t) = \rho'(\vec{x}',t) = \rho'(-\vec{x},t)$$

This makes sense because why should the charge density change if we just relabel the coordinates of a given point ?

Next we will transform Maxwell's equations to the primed system and by demanding that they remain invariant we will study how the fields and the current density change to that the equations remain invariant. Starting from (1) we have that

$$-\vec{\nabla}'\cdot\vec{E}(\vec{x},t) = 4\pi\rho(\vec{x},t) = 4\pi\rho'(\vec{x}',t) \Rightarrow$$

$$\vec{E}(\vec{x},t) = -\vec{E}'(\vec{x}',t) \Rightarrow \vec{E}'(\vec{x}',t) = -\vec{E}(\vec{x},t) \tag{11}$$

Also from (3) we have that

$$\vec{\nabla}'\times\vec{E}'(\vec{x}',t) = -\frac{\partial\vec{B}(\vec{x},t)}{c\,\partial t} \Rightarrow \vec{B}'(\vec{x}',t) = \vec{B}(\vec{x},t) \tag{12}$$

Similarly (4) gives

$$-\vec{\nabla}'\times\vec{B}'(\vec{x}',t) = \frac{4\pi}{c}\vec{J}(\vec{x},t) - \frac{\partial\vec{E}'(\vec{x}',t)}{c\partial t} \Rightarrow$$

$$\vec{J}'(\vec{x}',t) = -\vec{J}(\vec{x},t) \tag{13}$$

In conclusion, **Maxwell's equations are invariant under parity provided that the charge density is a scalar, the current density and the electric field are vectors and the magnetic field is an axial vector.**

From (5) and (12) we conclude that under parity the vector potential transforms as

$$\vec{A}'(\vec{x}',t) = -\vec{A}(\vec{x},t) \tag{14}$$

in other words it is a polar vector, and from (6) , (11) and (14) we can conclude that

$$\Phi'(\vec{x}',t) = \Phi(\vec{x},t) \tag{15}$$

in other words it is a scalar.

Charge Conjugation: Under charge conjugation all quantities change sign at any given space time point. That is

$$\rho'(\vec{x},t) = -\rho(\vec{x},t) \ \ , \ \ \vec{J}'(\vec{x},t) = -\vec{J}(\vec{x},t) \ \ , \ \ \vec{E}'(\vec{x},t) = -\vec{E}(\vec{x},t)$$

$$\vec{B}'(\vec{x},t) = -\vec{B}(\vec{x},t) \ \ , \ \ \vec{A}'(\vec{x},t) = -\vec{A}(\vec{x},t) \ \ , \ \ \Phi'(\vec{x},t) = -\Phi(\vec{x},t)$$

It is then trivial to see that **Maxwell's equations remain invariant under charge conjugation**.

Time Reversal: Under time reversal the charge density remains invariant. If equation (1) is to remain invariant then this means that

$$\vec{E}'(\vec{x},-t) = \vec{E}(\vec{x},t) \tag{16}$$

By requiring that equation (3) remains invariant we arrive to the conclusion that

$$\vec{B}'(\vec{x},-t) = -\vec{B}(\vec{x},t) \tag{17}$$

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Similarly, by requiring that equation (4) remains invariant we arrive to the current density transformation

$$\vec{J}'(\vec{x},-t) = -\vec{J}(\vec{x},t) \qquad (18)$$

From equation (5) and (17) we get that

$$\vec{A}'(\vec{x}',-t) = -\vec{A}(\vec{x},t) \qquad (19)$$

while from (6) and (19) we get that

$$\Phi'(\vec{x},-t) = -\Phi(\vec{x},t)$$

A summary of the results of this study is shown in Table 1.

	P	C	CP	T	CPT
$\rho(\vec{x},t)$	$+\rho(\vec{x},t)$	$-\rho(\vec{x},t)$	$-\rho(\vec{x},t)$	$+\rho(\vec{x},t)$	$-\rho(\vec{x},t)$
$\vec{J}(\vec{x},t)$	$-\vec{J}(\vec{x},t)$	$-\vec{J}(\vec{x},t)$	$+\vec{J}(-\vec{x},t)$	$-\vec{J}(\vec{x},t)$	$-\vec{J}(\vec{x},t)$
$\vec{E}(\vec{x},t)$	$-\vec{E}(\vec{x},t)$	$-\vec{E}(\vec{x},t)$	$+\vec{E}(\vec{x},t)$	$+\vec{E}(\vec{x},t)$	$+\vec{E}(\vec{x},t)$
$\vec{B}(\vec{x},t)$	$+\vec{B}(\vec{x},t)$	$-\vec{B}(\vec{x},t)$	$-\vec{B}(\vec{x},t)$	$-\vec{B}(\vec{x},t)$	$+\vec{B}(\vec{x},t)$
$\Phi(\vec{x},t)$	$+\Phi(\vec{x},t)$	$-\Phi(\vec{x},t)$	$-\Phi(\vec{x},t)$	$+\Phi(\vec{x},t)$	$-\Phi(\vec{x},-t)$
$\vec{A}(\vec{x},t)$	$-\vec{A}(\vec{x},t)$	$-\vec{A}(\vec{x},t)$	$+\vec{A}(\vec{x},t)$	$-\vec{A}(\vec{x},t)$	$-\vec{A}(\vec{x},t)$

Table 2: *Chance of the quantities involved in Maxwell's equation under P, C, T, CP and CPT.*

Hence, the equations of Maxwell are invariant under the combined CPT transformations. This is achieved by transforming the fields and current density as follows

$$F^{\mu\nu}(x^{\alpha}) \rightarrow F'^{\mu\nu}(x'^{\alpha}) = F'^{\mu\nu}(-x^{\alpha}) = F^{\mu\nu}(x^{\alpha})$$

$$A^{\mu}(x^{\alpha}) \rightarrow A'^{\mu}(x'^{\alpha}) = A'^{\mu}(-x^{\alpha}) = -A^{\mu}(x^{\alpha})$$

$$J^{\mu}(x^{\alpha}) \rightarrow J'^{\mu}(x'^{\alpha}) = J'^{\mu}(-x^{\alpha}) = -J^{\mu}(x^{\alpha})$$

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Searches for a non-vanishing Electric Dipole Moment of the Neutron (nEDM)

It is already well known that elementary spin ½ fermions exhibit a magnetic dipole moment and their interaction with a magnetic field via the dipole moment is described by

$$H_{\text{int}} = -\mu\vec{\sigma}\cdot\vec{B} \qquad (20)$$

where μ is the magnetic moment, $\vec{\sigma}$ is the spin and $\vec{B}$ is the magnetic field[1]. As we have seen, under Time Reversal both $\vec{\sigma}$ and $\vec{B}$ change signs. Hence, the above interaction term remains invariant under Time Reversal. Similarly under Parity both $\vec{\sigma}$ and $\vec{B}$ do not change sign (they are both axial vectors). **Hence, the magnetic dipole interaction term is invariant under both Parity and Time Reversal.**

Lets consider the possibility that a spin ½ fermion exhibits also an electric dipole moment. The spin of an elementary spin ½ fermion defines the only direction available in the reference frame of the fermion. Hence, the electric dipole moment, which transforms as a vector under rotations must be aligned with the spin and should point either to the direction of the spin or at the opposite direction. In other words

$$\vec{d} = d\frac{\vec{\sigma}}{|\vec{\sigma}|} \qquad (21)$$

where d is the hypothetical EDM of the fermion (the magnitude). If d is different than zero then the fermion can interact with an electric field via an interaction term of the form

$$H_{\text{int}} = -d\vec{\sigma}\cdot\vec{E} \qquad (22)$$

Under Time Reversal the spin changes sign. However, the electric field does not change sign under Time Reversal. Hence, such term is not invariant under Time Reversal. Similarly under Parity the spin does not change sign (axial vector) but the electromagnetic field does change sign (polar vector). **Therefore, a non-zero electric dipole moment would cause both Parity and Time reversal to be violated**.

There is another way of viewing this which is described in Figure 2. There it is shown how the spin and EDM transform under Parity and Time Reversal. In both cases we start from a fermion whose EDM is at the spin direction and we end up with a fermion whose dipole moment points opposite to the spin. If Time Reversal and Parity were conserved both types of particle should exist in nature.

1 See Homework Assignment 7.

In other words, if an elementary spin ½ fermion has a non-vanishing EDM and Parity and Time Reversal are good symmetries, then this fermion state should exist in nature in two degenerate states, one with the EDM at the direction to its spin and one which is opposite. It is clear that this is not the case. Hence, a discovery of a an elementary particle with non-zero EDM will be a clear sign that both Parity and Time Reversal are violated.

In conclusion, a non-vanishing EDM implies Parity and Time Reversal symmetry violation if the system has a non-degenerate ground state. As the reader may have noticed the origin of this violation is at the quantum level the relation (21) which forces the dipole moment, a polar vector, to transform like the spin which is an axial vector and this can be corrected only if the ground state is degenerate.

There are in nature molecules such as the H_2O or NH_3 which exhibit electric dipole moments. However, those do have degenerate ground states. Hence, their non zero EDM does not imply violation of either Parity or Time Reversal symmetries.

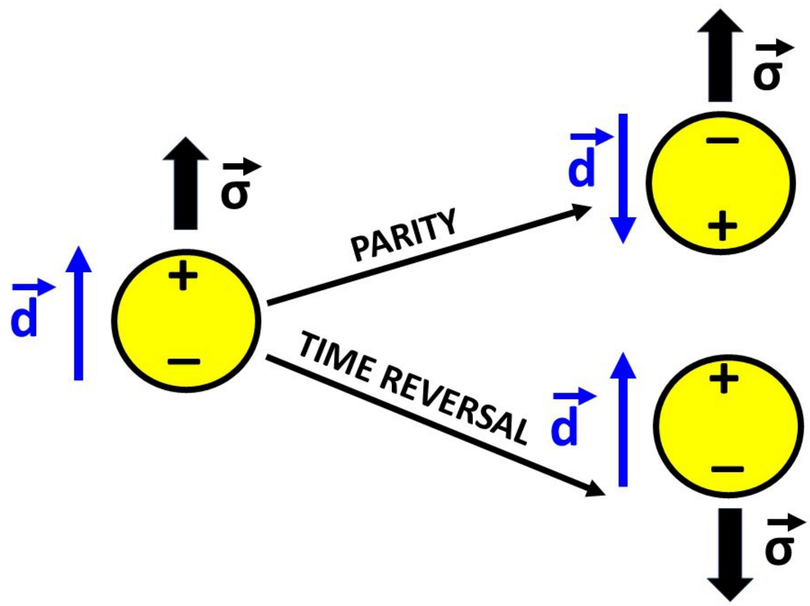

Figure 2: Transformation of the spin and EDM of a spin 1/2 fermion under Parity and Time Reversal.

Charge Parity Symmetry Violation in Particle Physics

The reader is probably wondering by now why the Time Reversal symmetry or its violation are so important. The reason is that if the Time Reversal Symmetry is violated, one concludes, using the CPT theorem, that also the CP symmetry is violated and the CP symmetry is intimately related to the question of the Baryon-Antibaryon Asymmetry in the Universe (BAU). In other words why our universe is made of matter and not antimatter given that during the Big Bang matter and antimatter must have been created in equal amounts. A. D. Sakharov, in a seminal publication in 1967, set three conditions which if satisfied can explain the BAU[2] . These were

1. Non-Conservation of the baryon number
2. CP violation
3. Non-thermodynamic equilibrium

Hence, establishing sources of CP violation will bring us closer to answering one of the most fundamental questions in physics today which is the BAU.

The Standard Model of particle physic includes two sources of CP violation:

The first comes from the Cabbibo-Kobayashi-Maskawa (CKM) matrix which determines the quark mixing and includes a phase δ which is non-zero and has as consequence that CP is violated. This sector of the Standard Model has been confirmed experimentally using both $K^0/\bar{K}^0$ and $B^0/\bar{B}^0$ data[3].

The second has its origin in the strong interaction sector. Quantum Chromodynamics (QCD), which is the theory of the strong interactions, and is part of the Standard Model, includes a term which is given by $L_{QCD}^{CPV} = \frac{g_s^2}{32\pi^2}\bar{\theta}\, G_{\mu\nu,a}\tilde{G}^{\mu\nu,a}$ where $\bar{\theta}$ is a phase, g_s is the strong coupling constant and $G_{\mu\nu,a}$, $\tilde{G}^{\mu\nu,a}$ are the gluon tensor and its dual[4]. $G_{\mu\nu,a}\tilde{G}^{\mu\nu,a}$ violates both Parity and Time Reversal symmetries[5]. So QCD predicts CP violation. However, despite all searches today, there has not been any experimental evidence that there are CP violating processes in strong interactions.

If one, using the CP phenomenoloy of the Standard Model, tries to predict the BAU, the result is a number which is eight orders of magnitude less than the number for BAU obtained from measurements of the cosmic microwave background radiation or measurements of the abundance of light elements produced in primordial nucleosynthesis[6].

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2 Andrej Sakharov (1967) Pisma Zh. Eksp. Teor. Fiz. 5 32; 1967 JETP Lett. 52 4; 1991 Sov. Phys.—Usp. 34 392; 1991 Usp. Fiz. Nauk 161 61

3 Add here some references from Kaon and B-meson experiments.

4 These are similar to the $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ Maxwell tensors of electromagnetism.

5 In Homework 11 a similar but simpler term is shown to violate Parity and Time Reversal symmetries.

6 A. Riotto and M. Trodden, Ann. Rev. Nucl. Part. Sci. 49, 35 (1999); M. Dine and A. Kusenko, Rev. Mod. Phys. 76, 1 (2003); R. H. Cyburt, B. D. Fields, and K. A. Olive, Cosmology Astropart. Phys. 11, 12 (2008) .

In other words we don't have an adequate theory that explains what happened to all antiparticles that must have been produced when the universe was created. **The CP violation processes in the Standard Model, allthough are in agreement with experimental data, cannot explain the BAU.** So there may be other, yet unkown, sources of CP violation. This is why it is important to investigate the question whether the neutron has a non vanishing EDM.

Estimates and Predictions of the Neutron Electric Dipole Moment

One can try to make some crude estimates of the magnitude of the nEDM. The most crude try would be to assume that the nEDM is the result of two opposite charges whose magnitude is equal to the magnitude of the electron charge and are positioned at a distance of 1 fm (approximately the size of a neutron). This would give an estimate of the order of

$$d_n \sim 10^{-13}\, e\, cm$$

We can do perhaps better than this if we assume that a possible non-zero nEDM is due to to the well established CP violation of the Weak Interaction. The coupling constat of the weak interaction has units of inverse-mass-square, GeV^{-2}, which is the same as length-square and is given by

$$G_F \sim \frac{g^2}{M_w^2} = \frac{4\pi\alpha_{QED}}{M_w^2}$$

where $\alpha_{QED} \approx 1/137$ is the coupling constant of the Electromagnetic interaction and $M_w \approx 80\, GeV$, the mass of the W-boson[7], the propagator of the charged current Weak Interaction. To estimate the nEDM we need somehow to derive a fundamental length from G_F which means that we have to multiply it with something that has units of mass. The only mass involved here is the neutron mass so we multiply the coupling constant with the mass of the neutron and finally we get

$$G_F \times m_n \sim \frac{4\pi\alpha_{QED}}{M_w^2} \times m_n = 1.4\times 10^{-5}\, GeV^{-1} \approx 10^{-19}\, cm$$

However, we know from Kaon data that the CP violating processes of the Weak interaction are further suppressed by another factor of $f \sim 10^{-3}$. Hence, we have that

[7]Here the assumption is that at very high energies (unification scale) the Weak and the Electromagnetic Interactions have the same strength but at lower energies, due to spontaneous symmetry breaking, the Weak interaction propagators acquire mass which the reason that the Weak interaction is suppressed relative to the Electromagnetic interaction.

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$$d_n \sim G_F \times m_n \times f \sim \frac{4\pi\alpha_{QED}}{M_w^2} \times m_n \times f \approx 10^{-22}\, cm$$

So our expectation is that the neutron EDM is incredibly small and as we will see later these estimates overestimate EDM by at least another four orders of magnitude.

Standard Model calculations at the 3d loop level (one and two loop calculations give null result for nEDM) based on the Kobayashi-Maskawa sector predict an even smaller nEDM[8].

$$d_n^{KM} \sim 10^{-32}\, e\, cm$$

From the CP violating sector of QCD one calculates that

$$d_n^{QCD} \sim \bar{\theta} \times 10^{-16}\, e\, cm$$

Taking in to account the current upper limit for the value of nEDM which is $d_n \sim 10^{-26} e\, cm$ this means that

$$\bar{\theta} < 10^{-10}$$

This is a surprising result and it is not yet known why $\bar{\theta}$ must be that close to zero since it is a phase and in principle could take any value between 0 and 2π. This is called the **Strong CP Problem** which can be solved by postulating the existence of spin zero particles called Axions. Although no Axions have ever been found searches for them continue. If they exist, except for solving the Strong CP problem, they are also a candidate for the missing dark matter of the universe.

Theories which are beyond the Standard Model predict a nEDM which is in the range between $10^{-17} - 10^{-28} e\, cm$. For example a prediction from Super Symmetry gives

$$d_n \sim \left(300 \frac{GeV}{\Lambda_{SUSY}}\right)^2 \times \sin(\varphi_{CP}) \times 10^{-24}\, e\, cm$$

where Λ_{SUSY} is the SUSY scale in GeV and φ_{CP} is a CP violating phase of SUSY.

8 Pospelov and Ritz, Ann. Phys. 318, 119, (2005)

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Measurements of nEDM

The fact that the predicted nEDM is so small poses serious challenge for experiments that endeavor to measure nEDM. Measurements of magnetic or electric dipole moments are usually done by measuring precession frequency of the magnetic or electric moment in a magnetic or electric field respectively as one can see from (20) and (22). However, due to the fact that the nEDM is so small one needs a very strong Electric field, of the order of 10 KV, and even then the precession frequency will be of the order of nHz[9]. For example assuming the current upper limit of the nEDM which is $d_n \sim 10^{-26} e\,cm$ and for an electric field equal to 15 KV the nEDM would precess and complete about two turns a year[10]. Even worse any small magnetic field of the order of fT leaking in the measurement region would cause the neuron, which has a magnetic moment, to precess with a similar frequency and thus fake the effect.

Experimenters have found a way to overcome these problems. As shown in Fig. 3 a fixed magnetic field $\vec{B}$ is introduced along with the electric field $\vec{E}$ and measurements of the precession angular frequency, ω, are made both with the electric field pointing at the direction of the magnetic field as well as pointing at the opposite direction. The magnetic field allows the neutron to precess with a higher and thus measurable frequency. As shown in Fig. 2 if the neutron has a non-zero EDM the difference between the energy levels when the electric field points to the direction of the magnetic field (left) should be larger than when it points to the opposite direction (right).

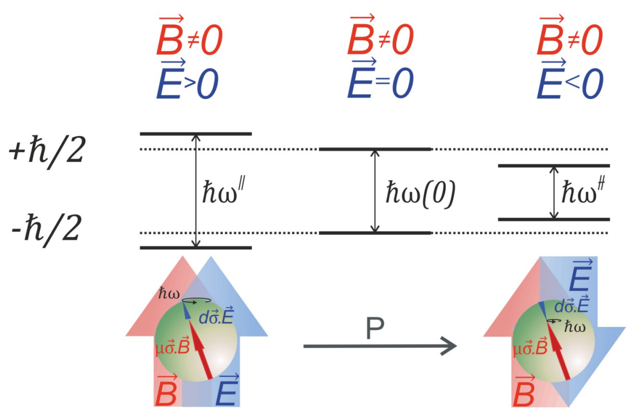

Figure 3: *Energy levels of a spin ½ fremion with a magnetic dipole moment and a hypothetical electric dipole moment. Measurements of nEDM use both a strong electric field and a magnetic field. Measurements are taken with the electric filed pointing at the direction of the magnetic field (left) and opposite to it (right).*

[9] P. Schmidt-Wellenburg, *The quest for an electric dipole moment of the neutron,* https://arxiv.org/abs/1602.01997

[10] Guillaume_Pignol, Philipp Schmidt-Wellenburg,*The search for the neutron electric dipole moment at PSI,* https://arxiv.org/abs/2103.01898

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The angular precession frequencies corresponding to the energy levels shown at the left and right of Fig. 3 are

$$\hbar\omega^{\uparrow\uparrow} = 2\left|\mu_n B^{\uparrow\uparrow} + d_n E^{\uparrow\uparrow}\right|$$

$$\hbar\omega^{\uparrow\downarrow} = 2\left|\mu_n B^{\uparrow\downarrow} - d_n E^{\uparrow\downarrow}\right|$$

By taking the difference between the angular frequency when the electric field points at the direction of the magnetic field with the angular frequency when the electric field points at the opposite direction the effect of the magnetic dipole moment cancels and one can measure nEDM.

$$d_n = \frac{\hbar(\omega^{\uparrow\uparrow} - \omega^{\uparrow\downarrow}) - 2\mu_n(B^{\uparrow\uparrow} - B^{\uparrow\downarrow})}{2(E^{\uparrow\uparrow} - E^{\uparrow\downarrow})}$$

and given that $E^{\uparrow\downarrow} = -E^{\uparrow\uparrow} = -E_0$ and $B^{\uparrow\downarrow} = B^{\uparrow\uparrow}$ we have that

$$d_n = \frac{\hbar(\omega^{\uparrow\uparrow} - \omega^{\uparrow\downarrow})}{4E_0}$$

So the problem of measuring nEDM is reduced to measuring the angular frequencies for the two configurations. The method for this was originally developed by N. F. Ramsey of Harvard[11] and the results from the experiment based on this method were published in 1957[12]. The apparatus of this experiment is shown in Figure 4.

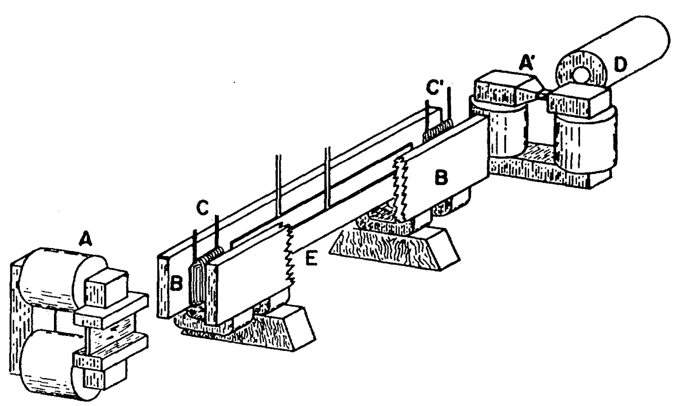

Figure 4: *Apparatus for measuring nEDM. A the magnetized iron mirror polarizer, A' the magnetized iron analyzer A', D the neutron detector. A uniform magnetic field is applied at B and each of the coils C and C' flip the neutron spin by π/2.*

[11] N. F. Ramsey, Phys. Rev. 78, 695, (1950).
[12] J. H. Smith, E. M. Purcell, N. F. Ramsey, Phys. Rev. 108, 120 (1957).

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In Figure 5 the procedure of measuring nEDM using this apparatus is illustrated.

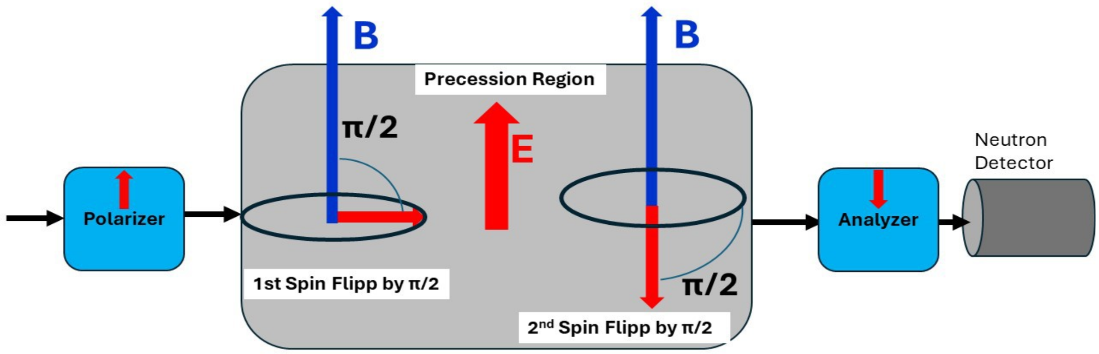

Figure 5: Neutron beam and the polarizer are shown at the left. The precession region with electric and magentic fileds E, B is shown at the center. The analyzer and the neutron detector are shown at right.

Neutrons with a Maxwellian velocity distribution corresponding to 500 K^0 produced by a reactor enter from the left and exit the polarizer with a specific polarization. Then they enter the precession region where a magnetic field B and an strong electric field E are applied and there their spins precess with an angular frequency ω_L . At the entrance and exist of the precession region an RF magnetic field B_{RF} perpendicular to B causes the neutron spin to flip by π/2. This spin-flip occurs only if the RF angular frequency ω_{RF} is equal to the precession angular frequency ω_L . The beam exists the precession region and enters the analyzer which selects neutrons whose polarization is opposite to the initial polarization. Neutrons existing the analyzer are counted by the neutron detector. Hence, the analyzer and the neutron detector essentially detect that the spin flip by π has occurred and this means that the externally supplied $\omega_{RF} = \omega_L$ and this way ω_L is measured.

The readers will find it perhaps interesting to discuss how the polarizer and the analyser select neutrons of a specific polarization. As is known from basic Quantum Mechanics moving neutrons can be described as de Broglie waves with wavelength $\lambda = h/p$ and when the cross media boundaries they undergo refraction just as the light waves. The index of refraction for neutrons entering from vacuum to a material is given by

$$\sqrt{1 - \frac{\lambda^2 N \alpha_{coh}}{\pi} \pm \frac{\mu_M B}{\frac{1}{2} M_n v^2}}$$

where λ is the neutron wavelength, N is the number of Nuclei per cubic centimeter, α_{coh} is the neutron forward scattering length, μ_M the neutron magnetic moment and B is the magnetic field. The sign $\pm$ refers to the orientation of the magnetic field relative to the magnetic moment of the neutron[13].

As it happens in optics neutron waves can undergo total reflection if the glancing angle θ is given by

$$\cos\theta_c = n$$

Hence, by appropriate choice of the material and the magnetic filed one can cause neutrons of a certain polarization to undergo total reflection. Hence, the neutrons that penetrate the magnetized material will have a specific polarization. The analyzer works exactly the same way.

The experiment found that the nEDM is equal to

$$d_n = (-0.1 \pm 2.4) \times 10^{-20}\, e\, cm$$

and from this they concluded that the nEDM must be

$$d_n < 5 \times 10^{-20}\, e\, cm$$

The resolution to these measurements is given by

$$\sigma(d_n) = \frac{\hbar}{2\alpha T E \sqrt{N}}$$

where N is the number of neutrons, T is the precession time, E is the electric field and α is a constant called *fringe visibility* and depends upon the neutron polarization and the analyzing power of the final neutron detector[14] .

As seen here, to make a precise measurement one need a long precession time and a large number of neutrons (experimental limitations do not allow to increase the Electric field significantly). Neutron beam experiments had short T and typically not a large number of neutrons so a way had to be found to collect a large number of neutrons and keep them for along time until they decay. This lead to the use of **Ultra Cold Neutrons** (UCN). These are neutrons with a velocity

$$v \sim 6\, m/s$$

13 N. F. Ramsey, *Electric Dipole Moment of the Neutron,* Annu. Rev. Nucl. Part. Sci. 1990. 40: 1-14

14 P. Schmidt-Wellenburg, *The quest for an electric dipole moment of the neutron,* https://arxiv.org/abs/1602.01997

which corresponds to a de Broglie wavelength and a temperature

$$\lambda = 0.066\,\mu m \qquad T \approx 2.2\,mK^o$$

respectively. Such UCN undergo total reflection at arbitrary glancing angle so they can be put in a 'bottle' and kept there to precess until they decay. Such experiments increase markedly the precession time and are thus more accurate than beam experiments.

The most recent measurement using UCNs comes from the experiment at Paul Scherrer Institute (PSI) in Villingen in Switzerland[15]. A diagram of the experimental apparatus is shown in Figure 6. UCN enter at bottom-left and are polarized by a 5T magnet and a spin flipper.

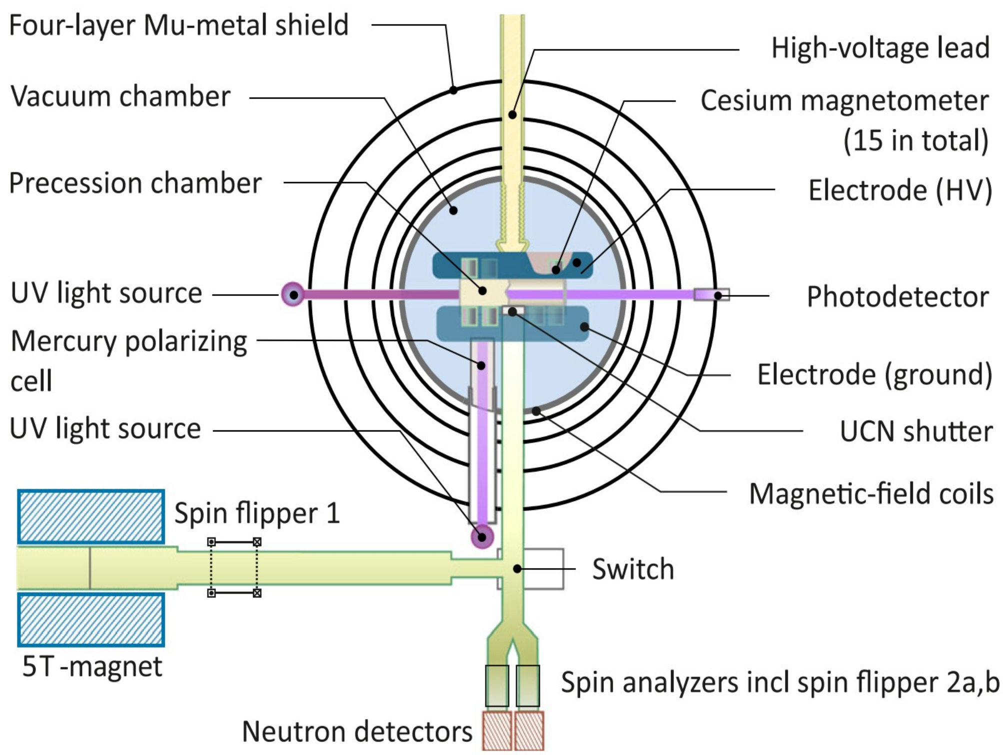

Figure 6: *The PCI experiment. The 5 Tesla magnet and the spin flipper is shown at the bottom left. Further to the right is the switch with directs the UCNs either to the precession chamber (center) or the the neutron detectors (bottom).*

[15] C. Abel et al., Phys. Rev. Lett. 124, 081803, (2020)

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The switch directs the neutrons to the cylindrical precession chamber (Radius R = 23.5 cm, Height H = 12cm). After a density of 2 UCN/cm^3 has been reached the UCN shutter at the bottom of the chamber is shut. The UCN in the chamber are subjected to an electric field equal to E = 11kV/cm and a collinear magnetic field equal to B = 1036nT. The UCNs are allowed to precess for 180 s and Ramsey's method was used to measure the precession angular frequency. At the end the shutter is opened and the switch directs the UCNs to the spin analyzers which measure both spin states. This way the neutron asymmetry

$$A = \frac{N\uparrow - N\downarrow}{N\uparrow + N\downarrow}$$

was measured for each time (cycle) that the chamber was filled with UCN, where $N\uparrow, N\downarrow$ are the numbers of neutrons with spin 'up' and 'down' respectively. In total 54068 such measurements were made each having on average 11400 UCNs. The measured nEDM was found to be

$$d_n = \left(0.0 \pm 1.1_{stat} \pm 0.2_{syst}\right)\times 10^{-26}\, e\, cm$$

which is consistent with a zero nEDM. As seen here the measurement is statistics limited. This result was interpreted to mean that if there is a non-vanishing nEDM then it must be, with 90% probability,

$$d_n < 1.8\times 10^{-26}\, e\, cm$$

and this constitutes the best limit today for nEDM.

New experiments are being constructed which aim to reach the sensitivity to observe an nEDM or set a limit at the level of $10^{-27} e\, cm$ which would make them sensitive to SUSY models. For example, the TRIUMF Ultra-Cold Advanced Neutron (TUCAN) collaboration has designed an experiment which will increase the UCN density to approximately 250 UCN/cm^3 or 10^6 per cycle and this, they claim, it will allow them to reach a sensitivity of $10^{-27} e\, cm$ in approximately 400 days[16]. Hence, not very long after the experiment starts collecting data, it will be sensitive to Physics Beyond the Standard Model.

[16] R. Matsumiya et al, The Precision nEDM Measurement with UltraCold Neutrons at TRIUMF, arXiv: 2207.09880v1, physics.ins-det, 19 Jul 2022.