Particle Physics Homework Assignment 7

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Problem 1: Consider $\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{u_A} \\ \boldsymbol{u_B} \end{pmatrix}$ to be solution of the Dirac equation where $\boldsymbol{u_A}, \ \boldsymbol{u_B}$ are two-component spinors. Show that in the non-relativistic limit where β is considerably smaller than 1, $\boldsymbol{u_B} \sim \boldsymbol{\beta} = \boldsymbol{v/c}$.

Problem 2: Show that at the non-relativistic limit the motion of a spin half fermion of charge $\boldsymbol{e}$ at the presence of an electromagnetic field $A^{\mu} = (A^0; \vec{A})$ is described by:

$$\left[\frac{(\vec{p}-e\vec{A})^2}{2m} - \frac{e}{2m}\vec{\sigma}\cdot\vec{B} + eA^0\right]\chi = E\chi$$

where $\vec{B}$ is the magnetic field, σ^i are the Pauli matrices and $E = p^0 - m$. Identify the g-factor of the fermion and show that the Dirac equation predicts the correct gyromagnetic ratio for the fermion. To write down the Dirac equation at the presence of an electromagnetic field substitute: $p^{\mu} \rightarrow p^{\mu} - eA^{\mu}$.

Problem 3: Show that:

(a) $\bar{\Psi}\gamma_5\Psi$ is a pseudoscalar.
(b) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi$ is an axial vector.

Comment on the Lorentz and parity properties of the quantities:

(a) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi\bar{\Psi}\gamma_{\mu}\Psi$
(b) $\bar{\Psi}\gamma_5\Psi\bar{\Psi}\gamma_5\Psi$
(c) $\bar{\Psi}\Psi\bar{\Psi}\gamma_5\Psi$
(d) $\bar{\Psi}\gamma_5\gamma^{\mu}\Psi\bar{\Psi}\gamma_5\gamma_{\mu}\Psi$
(e) $\bar{\Psi}\gamma^{\mu}\Psi\bar{\Psi}\gamma_{\mu}\Psi$

It is given that $\{\gamma_5, \gamma^{\mu}\} = \mathbf{0}$.

Problem 4: Let P be the parity operator acting on Dirac spinors such that:

$$P\,\Psi(x^{\mu}.\,p^{\mu}) = e^{i\varphi}\gamma^{0}\,\Psi(x^{\mu},p^{\mu})$$

Show that:

$$P\,\Psi^{(+)}(x^{\mu},p^{\mu}) = +\Psi^{(+)}(x^{\mu}{}',p^{\mu}{}')$$

and

$$P\,\Psi^{(-)}(x^{\mu},p^{\mu}) = -\Psi^{(-)}(x^{\mu}{}',p^{\mu}{}')$$

where $\Psi^{(+)}$, $\Psi^{(-)}$ are the positive and negative energy solutions of the Dirac equation and $x^{\mu} = (x^{0};\vec{x}),\ p^{\mu} = (p^{0};\vec{p}),\ x^{\mu}{}' = (x^{0};-\vec{x}) = (x^{0};\vec{x}\,')$,

$$p^{\mu}{}' = (p^{0};-\vec{p}) = (p^{0};\vec{p}\,')\ .$$

Problem 5: Consider the Dirac Hamiltonian:

$$\hat{H} = -i\,\vec{\alpha}\cdot\vec{\nabla} + \beta m\ .$$

Show that:

$$[\hat{H},\gamma^{0}\hat{\Pi}] = 0$$

where, $\hat{\Pi}$ is the coordinate parity operator such that $\hat{\Pi}\,f(\vec{r}) = f(-\vec{r})$.

Problem 6: Consider the Dirac equation for an electron which couples to the Electromagnetic field

$$\left[i\gamma^{\mu}(\partial_{\mu}-i\,e\,A_{\mu}(x))-m\right]\Psi(x) = 0$$

where $A^{\mu}(x) = (\Phi(x);\vec{A}(x))$ is the electromagnetic field. Show that the Dirac equation is invariant under parity provided that the electron and the electromagnetic field transform under parity as follows.

$$P\,\Psi(\vec{x},t) = \Psi'(\vec{x}\,',t) = \mathrm{e}^{i\varphi}\gamma^{0}\,\Psi(-\vec{x}\,',t)$$

$$P\,\Phi(\vec{x},t) = \Phi'(\vec{x}\,',t) = \Phi(-\vec{x}\,',t) = \Phi(\vec{x},t)$$

$$P\,\vec{A}(\vec{x},t) = \vec{A}'(\vec{x}\,',t) = -\vec{A}(-\vec{x}\,',t)$$