Particle Physics Homework Assignment 5

Prof. Costas Foudas, 09/11/22

Problem 1: As shown in class the Dirac matrices must satisfy the anti-commutator relationships:

$$\{\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j\} = 2\delta_{ij}, \quad \{\boldsymbol{\alpha}_i, \boldsymbol{\beta}\} = 0 \quad \text{with} \quad \boldsymbol{\beta}^2 = 1$$

I. Show that the $\boldsymbol{\alpha}_i$, $\boldsymbol{\beta}$ are Hermitian, traceless matrices with eigenvalues ± 1 and even dimensionality.

II. Show that, as long as the mass term is not zero and the matrix $\boldsymbol{\beta}$ is needed, there is no 2x2 set of matrices that satisfy all the above relationships. Hence, the Dirac matrices must be of dimension 4 or higher. First show that the set of matrices $(\mathbf{1}; \vec{\boldsymbol{\sigma}})$ can be used to express any 2×2 matrix. That is the coefficients c_0, c_i always exist such that any 2×2 matrix can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = c_0 \cdot 1 + c_i \cdot \sigma_i$$

Having shown this you can pick and intelligent choice for the $\boldsymbol{\alpha}_i$ in terms of the Pauli matrices, for example $\boldsymbol{\alpha}_i = \boldsymbol{\sigma}_i$ which automatically obeys $\{\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j\} = 2\delta_{ij}$, and express $\boldsymbol{\beta}$ in terms of $(\mathbf{1}; \vec{\boldsymbol{\sigma}})$ using (1) . Show then that there is no 2×2 $\boldsymbol{\beta}$ matrix that satisfies $\{\boldsymbol{\alpha}_i, \boldsymbol{\beta}\} = 0$.

Problem 2:

1. Show that the Weyl matrices:

$$\vec{\boldsymbol{\alpha}} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfy all the Dirac conditions of Problem 1. Hence, they form just another representation of the Dirac matrices, the Weyl representation, which is different than the standard Pauli-Dirac representation.

2. Show the Dirac matrices in the Weyl representation are

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3. Show that in the Weyl representation $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Problem 3: Use the Dirac Hamiltonian in the standard Pauli-Dirac representation,

$$H = \vec{\alpha}\cdot\vec{p} + \beta m$$

to compute $[H, \hat{L}]$ and $[H, \hat{\Sigma}]$ and show that they are not zero.
Use the results to show that:

$$[H, \hat{L} + (\frac{1}{2})\hat{\Sigma}] = 0$$

where the components of the angular momentum operator is given by:

$$\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$$

and the components of the spin operator are given by:

$$\hat{\Sigma}_i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

Recall that the Pauli matrices satisfy $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

Problem 4: Show that $(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0$