Particle Physics Homework Assignment 4

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Problem 1: Show the the momentum, $\boldsymbol{p}$ of a particle moving in a circular trajectory of radius, $\boldsymbol{R}$, in a magnetic field, $\boldsymbol{B}$, is given by

$$p = 0.3\, BR$$

where the momentum is given in **GeV/c** the magnetic field is in Tesla and the radius in meters.

Solution:

$$q\left(\vec{v}\times\vec{B}\right) = \frac{m\vec{v}^{\,2}}{R} \Rightarrow R = \frac{p}{qB}$$

where the momentum is given in **Kg m/sec** and the charge in **Cb.**

$$1\,eV = 1.6\,10^{-19}\,Cb\times 1\,Volt = 1.6\,10^{-19}\,Joules \Rightarrow 1\,GeV = 1.6\,10^{-10}\,Joules$$

$$1\,GeV/c = \frac{1.6\,10^{-10}}{3\,10^{8}}\,kg\,\frac{m}{sec} = \frac{1.6\,10^{-18}}{3}\,kg\,\frac{m}{sec}$$

Hence,

$$R[m] = \frac{p[GeV/c]\,\dfrac{1.6\,10^{-18}}{3}\,kg\,\dfrac{m}{sec}}{1.6\,10^{-19}\,Cb\,B[Tesla]} \Rightarrow R[m] = \frac{p[GeV/c]}{0.3\,B[Tesla]}$$

Problem 2: A particle of mass M and 4-momentum P decays into two particles of masses m_1 and m_2.

(1) Show that the energy of the 1st particle in the frame of M is:

$$E_1 = \frac{M^2+m_1^2-m_2^2}{2M}$$

(2) Show that the kinetic energy T_i of the ith particle in the frame of M is:

$$T_i = \Delta M\left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right)$$

where $\Delta M = M-m_1-m_2$ is the mass excess or Q value of the process.

(3) A lambda particle, Λ, is a neutral baryon of mass $M = 1115 \ MeV$ that decays into a proton of mass $m_1 = 938 \ MeV$ and pion of mass $m_2 = 140 \ MeV$. Compute the kinetic energies of the proton and the pion in the lambda rest frame.

Solution:

The kinetic energy is given by: $T_i = E_i^*-m_i$. Therefore if we can compute the energy of each particle we almost finished. But this we have done in class. At the M reference frame we have

$$P = (M;0), \ p_1 = (E_1^*;\vec{p}^*), \ p_2 = (E_2^*;-\vec{p}^*)$$

and $$P = p_1+p_2 \Rightarrow (P-p_1)^2 = p_2^2 \Rightarrow M^2+m_1^2-2P\cdot p_1 = m_2^2 \Rightarrow$$

$$E_1^* = \frac{M^2+m_1^2-m_2^2}{2M}$$

also

$$E_2^* = \frac{M^2+m_2^2-m_1^2}{2M}$$

$$T_1 = E_1^* - m_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} - m_1 = \frac{(M - m_1)^2 - m_2^2}{2M} \Rightarrow$$

$$T_1 = \frac{(M - m_1 - m_2)(M - m_1 + m_2)}{2M} = \frac{\Delta M}{2M}(M - m_1 + m_2)$$

$$T_1 = \frac{\Delta M}{2M}(2M - M - m_1 + m_2) = \Delta M\left(1 - \frac{M + m_1 - m_2}{2M}\right)$$

$$T_1 = \Delta M\left(1 - \frac{M - m_1 - m_2 + 2m_1}{2M}\right) \Rightarrow$$

$$T_1 = \Delta M\left(1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\right)$$

similarly

$$T_2 = \Delta M\left(1 - \frac{m_2}{M} - \frac{\Delta M}{2M}\right)$$

Ready to substitute the numbers:

$$\Delta M = 1115 - 938 - 140 \quad MeV = 37\, MeV$$

$$T_1 = 5.26\, MeV \quad \text{(proton)}$$

$$T_2 = 31.7\, MeV \quad \text{(pion)}$$

Problem 3: In a collision process a particle of mass $\boldsymbol{m}$ at rest in the laboratory, is struck by a particle of mass, $\boldsymbol{M}$, momentum, $\vec{P}_{LAB}$, and total energy, E_{LAB} .

(1) Show that: $W^2 = M^2 + m^2 + 2mE_{LAB}$ where W is the total energy at the center of mass system of the two particles.

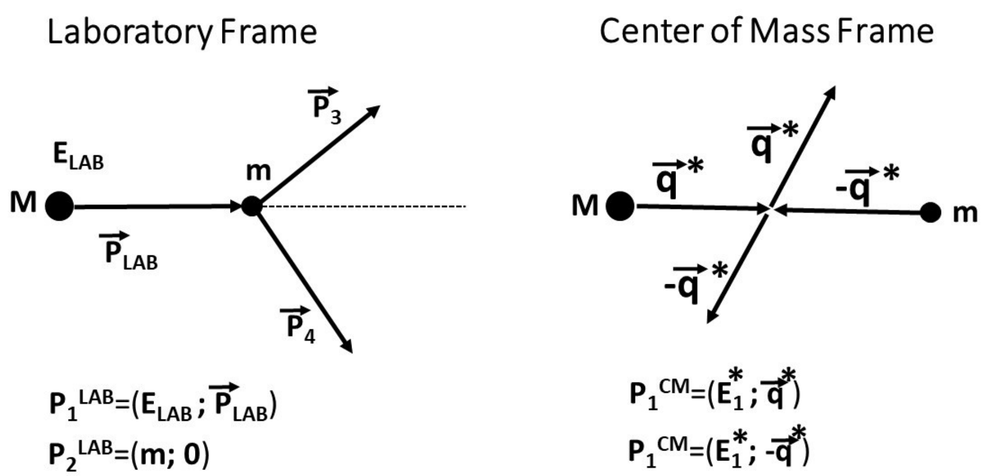
(2) Show that the momentum before the collision at the center of mass frame given by:

$$\vec{q}^{\,*} = \frac{m\vec{P}_{LAB}}{W}$$

(3) Show that the center of mass system of the two particles is boosted by:

$$\vec{\beta}_{cms} = \frac{\vec{P}_{LAB}}{m+E_{LAB}} \text{ and } \gamma = \frac{m+E_{LAB}}{W}$$

Solution:



(a) $W = E_1^* + E_2^* \Rightarrow W^2 = (P_1^{CM} + P_2^{CM})^2 = (P_1^{LAB} + P_2^{LAB})^2 = m^2 + M^2 + 2mE_{LAB} \Rightarrow$

$$W^2 = m^2 + M^2 + 2mE_{LAB}$$

(b)
$$|\vec{q}^*|^2 = (E_1^*)^2 - M^2 \tag{1}$$

$$P_1^{CM} + P_2^{CM} = (W;0) \Rightarrow P_1^{CM}\cdot(P_1^{CM}+P_2^{CM}) = P_1^{CM}\cdot(W;0) = W E_1^* \Rightarrow$$

$$P_1^{CM}\cdot(P_1^{CM}+P_2^{CM}) = P_1^{LAB}\cdot(P_1^{LAB}+P_2^{LAB}) = M^2 + m E_{LAB} = W E_1^* \Rightarrow$$

$$M^2 + mE_{LAB} = W E_1^* \Rightarrow E_1^* = \frac{(M^2 + m E_{LAB})}{W} \tag{2}$$

and from (1) and (2) we get

$$|\vec{q}^*|^2 = \left(\frac{M^2 + m E_{LAB}}{W}\right)^2 - M^2 = \frac{m^2 P_{LAB}^2}{W^2} \Rightarrow |\vec{q}^*| = \frac{m P_{LAB}}{W} \Rightarrow$$

(c)
$$P_1^{CM} + P_2^{CM} = (W;0) \tag{3}$$

$$P_1^{LAB} + P_2^{LAB} = (E_{LAB} + m; \vec{P}_{LAB}) \tag{4}$$

The boost at the x-direction for a 4-vector $A^\mu = (A^0; \vec{A})$ is given by

$$A^{0\prime} = \gamma(A^0 - \vec{\beta}\cdot\vec{A}) \tag{5}$$

$$A^{1\prime} = \gamma(A^1 - |\vec{\beta}| A^0) \tag{6}$$

$$A^{2\prime} = A^2$$

$$A^{3\prime} = A^3$$

Hence, from (3), (4) and (5) we have that

$$W = \gamma(E_{LAB} + m - \beta P_{LAB}) \tag{7}$$

and

$$0 = \gamma(P_{LAB} - \beta(E_{LAB} + m)) \Rightarrow \vec{\beta} = \frac{\vec{P}_{LAB}}{E_{LAB} + m} \tag{8}$$

Finaly from (8) and (7) we have that

$$W = \gamma\left(E_{LAB} + m - \frac{P_{LAB}^2}{E_{LAB}+m}\right) = \gamma \frac{W^2}{m+E_{LAB}} \Rightarrow \gamma = \frac{m+E_{LAB}}{W}$$

Problem 4: In an elastic scattering process the incident particle of mass M imparts energy to the stationary target of mass m. The energy ΔE lost by the incident particle appears as recoil kinetic energy of the energy of the target.

(1) Show that the kinetic energy transferred to the target particle at the lab frame is given by:

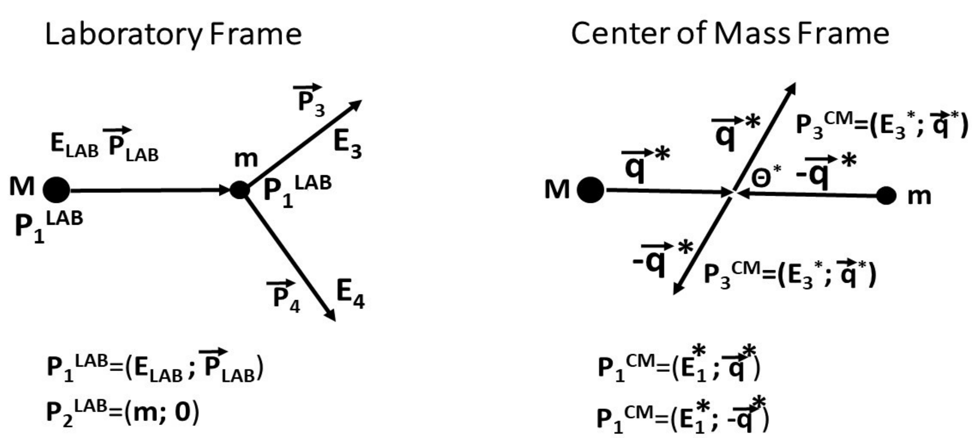
$$\Delta E = \frac{m P_{LAB}^2}{W^2}\left(1-\cos\theta^*\right)$$

where θ^* is the angle of the outgoing particles with respect the direction of the boost at the CM frame.

(2) Show that if $M \gg m$ the maximum kinetic energy of the recoil particle at the lab frame is:

$$\Delta E_{max} = 2\gamma^2 \beta^2 m$$

where β, γ are characteristic to the incident particle.



Solution:

We will use here all the formulas from the previous problem:

$$\vec{\beta}_{cms} = \frac{\vec{P}_{LAB}}{m+E_{LAB}} \qquad \text{(P3-1)}$$

and

$$\gamma_{cms} = \frac{m+E_{LAB}}{W} \qquad \text{(P3-2)}$$

(P3-1) and (P3-2) imply that:

$$\gamma_{CMS}\vec{\beta}_{CMS} = \frac{\vec{P}_{LAB}}{W} \qquad \text{(P3-3)}$$

The momentum of the target particle in CMS before the collision is:

$$\vec{P}^{*} = -\frac{m\vec{P}_{LAB}}{W} = -m\gamma_{CMS}\vec{\beta}_{CMS}$$

The energy of the target particle before the collision is:

$$E_2^{*} = \gamma_{CMS}\left(m-\vec{\beta}_{CMS}\cdot 0\right) = m\gamma_{CMS}$$

The kinetic energy transferred to the target particle is:

$$\Delta E = E_4^{LAB}-m \qquad (1)$$

So the idea is to find the energy of the target particle after the collision in the CMS frame and the boot back to the laboratory frame. In an elastic 2→2 process at the CMS we have

$$E_4^{*} = E_2^{*} \qquad ; \qquad |\vec{q}^{*}| = |\vec{P}^{*}|$$

Hence,

$$E_4^{LAB} = \gamma_{CMS}\left[m\gamma_{CMS}+\vec{\beta}_{CMS}\left(m\gamma_{CMS}|\vec{\beta}_{CMS}|(-\vec{q}^{*})\right)\right] \Rightarrow$$

$$E_4^{LAB} = m\gamma_{CMS}^2-m\beta_{CMS}^2\gamma_{CMS}^2\cos\theta^{*} \Rightarrow \qquad (2)$$

from (1) and (2) we have:

$$\Delta E \ = \ m\gamma_{CMS}^2 - m\,\beta_{CMS}^2\gamma_{CMS}^2\cos\theta^* \ - \ m \ = \ m(\gamma_{CMS}^2 - 1) - m\,\beta_{CMS}^2\gamma_{CMS}^2\cos\theta^* \ \Rightarrow$$

$$\Delta E \ = \ m\gamma_{CMS}^2\beta_{CMS}^2 - m\,\beta_{CMS}^2\gamma_{CMS}^2\cos\theta^* \ = \ m\gamma_{CMS}^2\beta_{CMS}^2(1-\cos\theta^*)$$

Using (P3-3) we get that:

$$\Delta E \ = \ \frac{m\vec{P}_{LAB}^2}{W^2}(1-\cos\theta^*)$$

Hence, the maximum kinetic energy is transferred when the recoil particle goes forward, i.e. when $\cos\theta^* \ = \ -1$. This result can be written as:

$$\Delta E_{MAX} \ = \ \frac{2m\vec{P}_{LAB}^2}{W^2} \ = \ \frac{2m\vec{P}_{LAB}^2}{M^2 \ + \ m^2 + 2mE_{LAB}} \ = \ \frac{2m\,\beta_{CMS}^2\gamma_{CMS}^2}{1 \ + \ \dfrac{m^2}{M^2} + \dfrac{2mE_{LAB}}{M^2}}$$

If $m^2 \ + \ 2mE_{LAB} \ll M^2$ (valid only for incident particles of low energy) then we obtain almost the formula often seen in books and used in calculating the Bethe-Bloch equation:

$$\Delta E_{MAX} \ = \ 2m\,\beta_{CMS}^2\gamma_{CMS}^2$$

Not exactly because the relativistic $\beta, \ \gamma$ here refer to the CMS boost where in Bethe-Bloch they refer to the incident particle. For this we need another approximation.

$$\vec{\beta}_{CMS} \ = \ \frac{\vec{P}_{LAB}}{E_{LAB}+m} \ = \ \frac{\vec{P}_{LAB}}{E_{LAB}}\left(1 \ + \ \frac{m}{E_{LAB}}\right)^{-1} \ = \ \frac{\vec{P}_{LAB}}{E_{LAB}}\left(1 \ - \ \frac{m}{E_{LAB}} \ + \ .....\right) \ \Rightarrow$$

$\vec{\beta}_{CMS} \ \simeq \ \vec{\beta}_1$ if $m \ \ll \ E_{LAB}$ which is usually the case. After this we have:

$$\Delta E_{MAX} \ = \ 2m\,\beta_1^2\gamma_1^2$$

Problem 5: A narrow pencil beam of singly charged particles of very high momentum p, traveling along the x-axis traverses a slab of material s radiation lengths in thickness. If ionization loss can be neglected, calculate the rms lateral spread of the beam in the y-direction, as it emerges from the slab. Use the formula you derive to compute the rms lateral spread of a beam of 10 GeV/c muons in traversing a 100 m pipe filled with (a) air (b) helium at NTP. The particle data book shows that the radiation length in air $X_0^{AIR} = 36.66\, g/cm^2$ and the density of air is $\rho^{AIR} = 1.293\, g/l$ where in Helium they are $X_0^{He} = 94.32\, g/cm^2$ and $\rho^{He} = 0.1786\, g/l$

Solution:

$$(d\theta_{PLANE})^2 = \left(\frac{21\, MeV}{\sqrt{2}\,\beta\, p[MeV]}\right)^2 \frac{dx}{X_0} \quad (1)$$

$$(dy)^2 = (d-x)^2 (d\theta_{PLANE})^2 \quad (2)$$

From (1) and (2) we have that:

$$(\Delta y)^2 = \int_0^d (dy)^2 = \left(\frac{21\, MeV}{\sqrt{2}\,\beta\, p[MeV]}\right)^2 \int_0^d \frac{(d-x)^2\, dx}{X_0} \Rightarrow$$

$$(\Delta y)^2 = \left(\frac{21\, MeV}{\sqrt{2}\,\beta\, p[MeV]}\right)^2 \frac{1}{X_0} \frac{d^3}{3} \Rightarrow$$

$$(\Delta y)^2 = \left(\frac{21\, MeV}{\sqrt{2}\,\beta\, p[MeV]}\right)^2 \frac{1}{X_0} \frac{(s X_0)^3}{3} \Rightarrow$$

$$(\Delta y)^2 = \left(\frac{21\, MeV}{\sqrt{2}\,\beta\, p[MeV]}\right)^2 \frac{s^3}{3} X_0^2 \Rightarrow$$

$$\Delta y = \frac{21\, MeV}{\sqrt{2}\,\beta\, p[MeV]} \frac{s^{\frac{3}{2}}}{\sqrt{3}} X_0 \quad (A)$$

So this is the general result and we will be using this to calculate the answers for this problem. First lets convert the radiation lengths of air and helium in centimeters:

$$X_0^{AIR}[cm] = \frac{36.66}{1.293\times10^{-3}} cm = 28350.5\, cm = 283.5\, m$$

$$X_0^{He}[cm] = \frac{94.32}{0.1786\times10^{-3}} cm = 528107.5\, cm = 5281\, m$$

Muons at 10 GeV are relativistic particles so they have $\beta \simeq 1$. So now we have everything we need to to compute the beam spread in centimeters. From (A) we have that

$$\Delta y^{AIR} = 5.1\, cm \quad \text{and} \quad \Delta y^{He} = 1.2\, cm$$

Problem 6: A sampling electromagnetic calorimeter uses **1.5 mm** thick iron plates as an the absorber spaced by **2.0 mm** layers of liquid Argon (originally developed by Willis and Radeka). The MIP values for Argon and Iron are $\mathbf{1.519\,MeV\,g^{-1}\,cm^2}$ and $\mathbf{1.451\,MeV\,g^{-1}\,cm^2}$ respectively. The density of Argon is $\rho^{Ar} = 1.396\,g/cm^3$ and that of iron is $\rho^{Fe} = 7.870\,g/cm^3$. Iron has atomic number $Z^{Fe} = 26$, an average mass number of $A^{Fe} = 55.85$ and $X_0^{Fe} = 1.76\,cm$.

a) Compute the sampling fraction of this calorimeter. Assume that after the electromagnetic shower is formed all charged particles behave as minimum ionizing particles and compute the energy losses using the Bethe-Bloch equation.
b) Compute the critical energy for the absorber of the calorimeter.
c) Estimate the coefficient of the stochastic term of the calorimeter resolution if the energy cut off correction is $F = 0.869$.
d) Compute the number of sampling elements (pairs of absorber and active material) required to contain at least 95% of showers coming from electrons of energy ***30 GeV***.
e) Suppose that electrons illuminate the calorimeter. As electron strike the front face of the calorimeter one has to take in to account edge effects. That is energy miss-measurement due to electrons whose shower escapes transversely from the sides of the calorimeter. Assuming that the impact point of the electrons at the front face is known (for example via some tracking device placed in front of the calorimeter) how far does this impact point need to be from the edge of the calorimeter so that the shower is also 95% contained also in the transverse direction ?

Solution:

(a) The sampling ratios is the ratio of the energy deposition of a minimum ionizing particle in the active material over the energy deposition of the same particle in the absorber:

$$f = \frac{d_{Act}\left(\frac{dE}{dx}\right)_{Act}}{d_{Act}\left(\frac{dE}{dx}\right)_{Act} + d_{Abs}\left(\frac{dE}{dx}\right)_{As}} \Rightarrow$$

$$f = \frac{0.2\,cm\;\;1.519\,MeV\,g^{-1}cm^2\;\;1.396\,g\,cm^{-3}}{0.2\,cm\;\;1.519\,MeV\,g^{-1}cm^2\;\;1.396\,g\,cm^{-3} + 0.15\,cm\;\;1.451\,MeV\,g^{-1}cm^2\;\;7.87\,g\,cm^{-3}}$$

$$f = 0.198 \Rightarrow f \simeq 20\%$$

Hence, 20% of the energy is seen in the active material and the other is absorbed in the absorber.

(b) The critical energy for iron can be calculated using the formula:

$$E_c = \frac{610\,MeV}{Z + 1.24} = \frac{610\,MeV}{26 + 1.24} = 22.39\,MeV$$

(c) The stochastic term of the resolution is given by:

$$\frac{\sigma(E)}{E} = \sqrt{\frac{E_c d_{Fe}}{f\,E\,X_0 F}} = \sqrt{\frac{22.39\,MeV \quad 0.15\,cm}{0.2 \quad 1000\,MeV \quad 1.76\,cm \quad 0.869 \quad E[GeV]}} = \frac{0.105}{\sqrt{E[GeV]}} \Rightarrow$$

$$\frac{\sigma(E)}{E} = \frac{10.5\,\%}{\sqrt{E[GeV]}}$$

Notice that

$$\frac{E}{E_c \times \frac{d_{Fe}}{X_0}}$$

is simply the number of particles in the shower corresponding to a given energy. The factor $f = \frac{E_{vis}}{E}$ is the fraction of the visible energy by this calorimeter over the incident energy. That is the energy deposited on the active material. In this process we lose some particles because the energy they deposit is below a certain threshold and they get absorbed without interacting with the active material. The effect of this is described by the factor F .

(d) As we have seen in the course the number of radiation lengths to contain the shower increases logarithmically (base e) with energy. As we have seen in the course in a calorimeter a new generation of particles is crated for every X_0 that the particles traverse until the reach critical energy and start loosing energy via the ionization process.

$$t_{max} = \ln\left(\frac{E}{E_c}\right) - 0.5[X_0] = \ln\left(\frac{3\times10^4}{22.39}\right) - 0.5 \quad [X_0] = 6.7\,X_0 = 11.8\,cm$$

So after 79 sampling elements we have shower maximum.

To contain the shower to 95% level we need:

$$t_{95\%} = t_{max} + 0.08 \times Z + 9.6 \; [X_0] = 18.4\,X_0 = 32.4\,cm$$

Hence, after 216 sampling elements we have contained 95% of the shower. However in practice experiments wish to contain if possible all the shower and thus achive the best possible resolution. So they use calorimeters with a depth of about $26\,X_0$ or more.

(e) One can estimate the Moliere radius from:

$$R_M \simeq \frac{21\,MeV}{E_c}[X_0] = 1.65\,cm$$

or from :

$$R_M = \frac{7A}{Z}[g\,cm^{-2}]$$

where one has to divide by the density of iron to get:

$$R_M \simeq \frac{7A}{Z}(\frac{1}{\rho^{Fe}})cm = \frac{7\times 55.85}{26\times 7}cm = 1.9\,cm$$

which is not very far from the previous result. Students should get from this an idea about the accuracy of these estimates.

Typically as we have learned we get 95% transverse containment if we go a disance of twice the Molier radius from the shower center. Hence,

$$R_{95\%} \simeq \frac{14A}{Z}(\frac{1}{\rho^{Fe}})cm = \frac{14\times 55.85}{26\times 7}cm = 3.8\,cm$$

In other words: The energy measurement, for electrons that have an impact point about ***4 cm*** from the edges of the calorimeter, is incorrect because part of the shower escapes in the transverse direction.